Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 53-59

Review of Research
Faculty of Science
Mathematics Series

NORMAL FUZZY SUBGROUPS

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Abstract

Rosenfeld in [8] has introduced the concept of a fuzzy subgroup. His definition has been generalised by Anthony and Sherwood in [2]. In this paper we introduce the concept of a normal fuzzy subgroup and prove some properties of this new concept.

AMS Mathematics Subject Classification (1980): 20N25
Key words and phrases: Normal fuzzy subgroups

Throughout this paper G will denote a group whose operation is suppressed and indicated by juxtaposition. We use the concept of a fuzzy subgroup as defined in [2]. For the definition of a t - norm we refer the reader to [1, 2]. We do not distinguish in notion between a fuzzy set [11] and its membership function.

Definition 1. *Let* A *be a fuzzy subset of* G *and let* ${}_xf : G \longrightarrow G$ $(f_x : G \longrightarrow G)$ *be a function defined by* ${}_xf(g) = xg$ $(f_x(g) = gx)$. *A fuzzy left (right) coset* $xA(Ax)$ *is defined to be* ${}_xf(A)\,(f_x(A))$.

If is easily seen that $(xA)(y) = A(x^{-1}y)$ and $(Ax)(y) = A(yx^{-1})$, for every y in G.

Proposition 1. *Let A be a fuzzy subset of G. Then the following conditions are equivalent for each x, y in G.*
(i) $A(xyx^{-1}) \geq A(y)$.
(ii) $A(xyx^{-1}) = A(y)$.
(iii) $A(xy) = A(yx)$.
(iv) $xA = Ax$.
(v) $xAx^{-1} = A$.

Proof. Straightforward.

Proposition 2. *If A is a fuzzy subgroup of G, then gAg^{-1} is also a fuzzy subgroup of G for all g in G.*

Proof. Let A be a fuzzy subgroup of G, under a t - norm T. Then $gAg^{-1}(e) = A(e) = 1$, $gAg^{-1}(x) = A(g^{-1}xg) = A(g^{-1}xg) = gAg^{-1}\ (x^{-1})$ for all x, g in G and $gAg^{-1}\ (xy) = A(g^{-1}(xy)g) = Ag^{-1}(xgg^{-1}y)g) = A((g^{-1}xg)(g^{-1}yg)) \leq T(A(g^{-1}xg),\ Ag^{-1}yg)) = T(gAg^{-1}\ (x),\ gAg^{-1}\ (y))$ for all x, y in G. Hence gAg^{-1} is a fuzzy subgroup of G under T.

Definition 2. *A fuzzy subgroup A of G is said to be normal if it satisfies one of the equivalent conditions in Proposition 1.*

Proposition 3. *If A is a fuzzy subgroup of G, then $\bigcap_{g \in G} gAg^{-1}$ is a normal fuzzy subgroup of G (under a t - norm as A).*

Proof. Similar to the crisp case.

Proposition 4. *Let A and B be two fuzzy subgroups of G under the t - norms T_1 and T_2, respectively. Then $A \cap B$ is a fuzzy subgroups under any t- norm T such that $T_1, T_2 \geq T$.*

Proof. Let $X = \{T | T_1, T_2 \geq T\}$ and let Z be a t - norm defined by:

$$Z(a,b) = 0 \text{ if } a \neq 1, b \neq 1 \text{ and } Z(a,b) = a (= b) \text{ if } b = 1\ (a = 1).$$

Since $Min \geq T' \geq Z$, for any t - norm T' (see [6, 10]), then $Z \in X$, i. e., $X \neq \emptyset$. Now let x, y in G. Since $Min \gg T' \gg T' \gg Z$ (see [10]); then

$A \cap B\,(xy^{-1}) = Min\,(A\,(xy^{-1}), B\,(xy^{-1})) \geq Min(T_1\,(A\,(x),\; A\,(y)),$
$T_2\,(B\,(x), B\,(y)) \geq Min\,(T\,(A\,(x),\; A(y)),\; T\,(B\,(x),\; B\,(y))) =$
$= T\,(Min\,(A\,(x),\; B\,(x)),\; Min\,(A\,(y),\; B\,(y))) =$
$= T\,(A \cap B\,(x),\; A \cap B\,(y)).$
Hence $A\,B$ is a fuzzy subgroups of G under T.

Proposition 5. *The intersection of any two normal fuzzy subgroups of G is also a normal fuzzy subgroup of G under any t - norm weaker than the t - norms of the two fuzzy subgroups.*

Proof. Let A and B be two normal fuzzy subgroups of G under t - norms T and T', respectively. According to Proposition 4, $A \cap B$ is a fuzzy subgroup of G under any t - norm weaker than T and T'. Now, for all x, y in G we have $A \cap B(xyx^{-1}) = Min(A(xyx^{-1}), B(xyx^{-1})) = Min(A(y), B(y)) = A \cap B(y)$. Hence $A \cap B$ is a normal fuzzy subgroup of G.

Proposition 6. *Let $f : G \longrightarrow H$ be a group homomorphism.*

(i) If A is a normal fuzzy subgroups of H, then $f^{-1}(A)$ is a normal fuzzy subgroup of G (under the t - norm of A).

(ii) If f is an epimorphism and A is a normal fuzzy subgroups of G, then $f(A)$ is a normal fuzzy subgroups of H (under the t - norm of A, which is continious).

Proof.

(i) Let A be a fuzzy subgroup of H under T. Then $f^{-1}(A)$ is a fuzzy subgroup of G under T (cf. [2,3]). Now, for all x, y in G, we have $f^{-1}(A)(xyx^{-1}) = A(f(xyx^{-1})) = A(f(x)f(y)f(x)^{-1}) = A(f(y)) = f^{-1}(A)(y)$. Hence $f^{-1}(A)$ is a normal fuzzy subgroup of G.

(ii) Let A be a fuzzy subgroup of G under a continous t - norm T. Then $f(A)$ is a fuzzy subgroup of H under T (cf. [2]). Now, for all u, v in H, we have

$$f(A)(uvu^{-1}) = \sup_{F(y)=uvu^{-1}} A(y) = \sup_{F(x)=u, f(y)=v} A(xyx^{-1})$$
$$= \sup_{F(y)=v} A(y) = f(A)(v), \text{ since } f \text{ is epimorphism.}$$

Hence $f(A)$ is a normal fuzzy subgroup of H.

Definition 3. [9]. *Let A and B be two fuzzy subsets of G. The T - product of A and B is defined to be the fuzzy subset AB of G given by:*

$$AB(x) = \sup_{yz=x} T(A(y), A(z)), \quad x \in G.$$

Note that a fuzzy coset xA may be also defined as the T -product of $\{x\}$ and A.

Proposition 7. *Let $f : G \longrightarrow H$ be a group homomorphism and let A and B a fuzzy subsets of G. Then $f(AB) = f(A)f(B)$, where the products are under the same t - norm.*

Proof. Similar to the proof of Proposition 2.2 in [5].

Note that Proposition 6(ii) may also be proved, using Proposition 7 as follows: Since $xAx^{-1} = A$ for all $x \in G$; then $f(xAx^{-1}) = f(A)$, i. e., $f(x)f(A)f(x)^{-1} = f(A)$. But since f is an epimorphism, then $f(A)$ is a normal fuzzy subgroup of H.

Proposition 8. [3]. *Let A be a fuzzy subgroup of G under Min. Then the level subset $G^t = \{x \in G \mid A(x) \geq t\}$, for every t in [0, 1] such that $G^t_A \neq \emptyset$, is a subgroup of G.*

In the above proposition, Min can not be replaced by an arbitrary t - norm. The following example indicates this fact.

Example 1. *Let $G = \{e, b, c, d\}$ be a Klein's four group and let A be a fuzzy subset of G defined by $A(e) = 1$, $A(b) = t_1$, $A(c) = t_2$ and $A(d) = t_3$ such that $t_3 = t_1 t_2$ and $1 \geq t_1 \geq t_2 \geq t_3$. Then it is easily verified that A is a fuzzy subgroup of G under Prod. But $G^{t_2}_A$ is not a subgroup of G; since $bc = d \notin G^{t_2}_A = \{a, b, c\}$.*

Proposition 9. .

(i) If A is a normal fuzzy subgroup of G under Min, then every G^t_A $(\neq 0)$, t in [0, 1], is a normal subgroup of G.

(ii) Let A be a fuzzy subset of G such that, for every t in $Im(A)$, G is a normal subgroup of G. Then A is a normal fuzzy subgroup of G under Min.

Proof. From Proposition 8, it follows that G_A^t, for every t in [0, 1], is a subgroup of G. Let y in G. Then $A(xyx^{-1}) = A(y) \geq t$. Thus $xyx^{-1} \in G_A^t$, i. e., $y \in x^{-1}G_A^t x$. This implies that G is a normal subgroup of G.

(ii) Let G_A^t, t in Im (A), be a normal subgroup of G. Then $xG_A^t\ x^{-1} \subseteq G_A^t$, for all $x \in G$. Let $y \in G$ with $A(y) = t$. Then $A(xyx^{-1}) \geq t = A(y)$. Since A is a fuzzy subgroup (under Min); then A is a normal subgroup of G.

One may easily verify the following.

Proposition 10. *If A and B are normal fuzzy subgroups of G, then the T - product AB is a normal fuzzy subgroup of G under T.*

Definition 4. *Let A be a fuzzy subgroup of G and M be a subset of G such that $mAm^{-1} = A$, for all $m \in M$. We call M the normaliser of A.*

Proposition 11. *The normaliser of a fuzzy subgroup of G is a subgroup of G.*

Proof. Similar to the crisp case.

Note that, if A is a fuzzy subgroup of G and M its normaliser, then $G_A^1 \subseteq M$. Note also that a fuzzy subgroup A of G is a normal iff its normaliser $M = G$.

Example 2. *Let $G = \{a, b, c, d, e, f\}$, where $a = (1)$, $b = (12)(36)(45)$, $c = (13)(25)(46)$, $d = (14)(26)(35)$, $e = (156)(234)$ and $f = (165)(243)$.*

1. If A is a fuzzy subset of G defined by : $A(a) = 1$, $A(b) = A(c) = A(d) = t_1$ and $A(e) = A(f) = t_2$, such that $1 \geq t_2 \geq t_1$, where t_1 and t_2 in [0,1]. Then A is a fuzzy subgroup of G under Min and $G = M$ (M is the normaliser of A). Hence A is normal.

2. If A is a fuzzy subset of G defined by : $A(a) = 1$, $A(b) = t_1$, $A(c) = A(d) = A(e) = A(f) = t_2$, such that $1 \geq t_2 \geq t_1$, where t_1 and t_2 in [0, 1]. Then A is a fuzzy subgroup of G under Min and its normaliser $M = \{a, b\}$. Note that if $t_2 \neq 0$, then $A \not\subseteq M$.

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REZIME

NORMALNE RASPLINUTE PODGRUPE

Rosenfeld je u [8] uveo pojam rasplinute podgrupe. Njegovu definiciju su uopštili Anthony i Sherwood u [2]. U ovom radu je uveden pojam normalne rasplinute podgrupe i ispitane su neke osobine ovog novog pojma.

Received by the editors October 10, 1986.