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ON MULTIVALUED CONTRACTIONS IN PROBABILISTIC METRIC SPACES

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Abstract

Some properties of multivalued contractions in probabilistic metric spaces are investigated.

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1. Introduction

For singlevalued mappings there are two types of contractions in probabilistic metric spaces. The first one is introduced by V.H.Sehgal and A.T.Bharucha-Reid in [9] and the second one by T.L.Hicks in [5].

Definition 1. *A mapping $f : S \to S$, where $(S, \mathcal{F})$ is a probabilistic semi-metric space, is a $B-$contraction if there is a $k \in (0,1)$ such that for every $p, q \in S$ and all $x > 0$*

$$F_{fp,fq}(kx) \geq F_{p,q}(x).$$

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Definition 2. *A mapping $f : S \to S$, where $(S, \mathcal{F})$ is a probabilistic semimetric space, is an $H-$contraction if there is a $k \in (0,1)$ such that for every $p, q \in S$ and every $x > 0$*

$$F_{p,q}(x) > 1 - x \Rightarrow F_{fp,fq}(kx) > 1 - kx.$$

Let $(S, \mathcal{F})$ be a probabilistic semimetric space and for every $p, q \in S$

$$\beta(p,q) = \inf\{h; F_{p,q}(h^+) > 1 - h\}.$$

If $(S, \mathcal{F}, t)$ is a probabilistic metric space with $t \geq t_m$ $(t_m(x,y) = \max\{x + y - 1, 0\}, (x,y) \in [0,1]^2)$ then β is a metric on S [6] .

In [6] the following Theorem is proved .

Theorem A. *The mapping $f : S \to S$ is an $H-$contraction on probabilistic metric space $(S, \mathcal{F}, t)$ with $t \geq t_m$ if and only if f is a metric contraction on the metric space (S, β).*

A $B-$contraction need not be an $H-$contraction and an $H-$contraction need not be a $B-$contraction [8].

In [8] some sufficient conditions for $\mathcal{F}$ are given such that every B contraction on $(S, \mathcal{F})$ is an $H-$contraction.

Theorem B. *Let $(S, \mathcal{F})$ be a probabilistic semimetric space such that $Ran(\mathcal{F})$ is finite and $Ran(\mathcal{F}) \setminus \{\epsilon_0\}$ is strictly increasing on $[0,1]$, where $\epsilon_0(x) = 0$ for $x \leq 0$ and $\epsilon_0(x) = 1$ for $x > 0$. Then every $B-$contraction on $(S, \mathcal{F})$ is an $H-$contraction.*

In general, it is not true that every $B-$contraction is an $H-$contraction and some examples are given in [8].

For multivalued mappings up to now there are three definitions of multivalued probabilistic contractions which are in some sense generalization of Definitions 1 and 2. The aim of this paper is to give some properties of a multivalued probabilistic contraction with respect to Definitions 3, 4 and 5 given below.

2. Preliminaries

Let $(S,\mathcal{F})$ be a propabilistic semimetric space and A a nonempty subset of S. The function $D_A(\cdot)$, defined by

$$D_A(u) = \sup_{s<u} \inf_{p,q\in A} F_{p,q}(s), \quad u \in \mathbf{R}^+$$

is called the probabilistic diameter of the set A and the set A is probabilistic bounded if and only if

$$\sup_{u\in\mathbf{R}^+} D_A(u) = 1.$$

For every two probabilistic bounded subsets A and B from S

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s).$$

Definition 3. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space, $f : S \to nB(S)$ (nonempty, bounded subsets of S) and there exists $k \in (0,1)$ such that*

$$\tilde{F}_{fp,fq}(ku) \geq F_{p,q}(u), \quad \textit{for every } p,q \in S \textit{ and every } \quad u > 0.$$

Then f is a B – contraction type mapping .

Definition 4. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space, $f : S \to n(S)$ and there exists $k \in (0,1)$ such that the following condition is satisfied:*

For every $p,q \in S$ and for every $v \in fp$
there exists $w \in fq$ such that for every $u > 0$
$$F_{v,w}(ku) \geq F_{p,q}(u).$$

Then f is a C – contraction type mapping.

Definition 5. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space, $f : S \to n(S)$ and there exists $k \in (0,1)$ such that the following implication holds for every $p,q \in S$ and $u > 0$:*

$$F_{p,q}(u) > 1 - u \Rightarrow \tilde{F}_{fp,fq}(ku) > 1 - ku.$$

Then f is an H_1 – contraction type mapping.

Definition 6. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space, $f : S \to n(S)$ and there exists $k \in (0,1)$ such that the following implication holds for every $p, q \in S$ and $x > 0$:*

$$F_{p,q}(x) > 1 - x \Rightarrow \textit{for every } u \in fp \textit{ there exists } v(u) \in fq \textit{ such that } F_{u,v(u)}(kx) > 1 - kx.$$

Then f is an H_2 - contraction.

The Hausdorff function of noncompactness $\beta_A(\cdot)$ (A is a probabilistic bounded subset of S) is defined in the following way [10]:

$$\beta_A(u) = \sup\{\epsilon > 0; \quad \text{there exists a finite subset } A_f \text{ of } S \text{ such that } \tilde{F}_{A,A_f}(u) \geq \epsilon\}.$$

The function β has the following properties:

(i) $\beta_A \in \Delta$, where Δ is the set of distribution functions.

(ii) $\beta_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}$.

(iii) $\emptyset \neq A \subset B \subset S \Rightarrow \beta_A(u) \geq \beta_B(u)$, for every $u \in \mathbf{R}$.

(iv) $\beta_{A\cup B}(u) = \min\{\beta_A(u), \beta_B(u)\}$, for every $u \in \mathbf{R}$.

(v) $\beta_A(u) = \beta_{\bar{A}}(u)(u \in \mathbf{R})$, where $\bar{A}$ is the closure of A.

(vi) $\beta_A = \epsilon_0 \Rightarrow A$ is precompact.

If $(S,\mathcal{F})$ is a probabilistic semimetric space , K a probabilistic bounded subset of S and T a mapping from K into $n(S)$ (nonempty subsets of S) we say that T is densifying on K with respect to β if $T(K)$ is probabilistic bounded and for every $B \subset K$:

$$\beta_{T(B)}(u) \leq \beta_B(u) \quad \text{foe every } u > 0 \Rightarrow B \quad \text{is precompact.}$$

A mapping $f : K \to n(S)$ is a k - set probabilistic contraction if $T(A) \in B(S)$, for every $A \subseteq K$ and

$$\beta_{T(A)}(ks) \geq \beta_A(s), \quad \text{for every} \quad s > 0.$$

The first result is a generalization of Theorem 3.1 from [8] .

Theorem 1. *Let $(S,\mathcal{F})$ be a probabilistic semimetric space and $f : S \to Com(S)$ (nonempty compact subsets of S) be a $B-$ contraction type mapping. If $Ran(\mathcal{F})$ is finite and each element of $Ran(\tilde{\mathcal{F}}) \setminus \{\epsilon_0\}$ is strictly increasing on $[0,1]$ then there exists $\gamma \in (0,1)$ such that*

$$D_\beta(fp, fq) \leq \gamma\beta(p,q), \quad \textit{for every } \ p,q \in S$$

where D_β is defined by:

$$D_\beta(A,B) = \max\{\sup_{p\in A}\inf_{q\in B}\beta(p,q), \sup_{p\in B}\inf_{q\in A}\beta(p,q)\}.$$

First, we shall prove the following Lemma.

Lemma 1. *Let $(S,\tilde{\mathcal{F}})$ be a probabilistic semimetric space and $f : S \to Com(S)$ be a B – contraction type mapping such that each element of $Ran(\tilde{\mathcal{F}}) \setminus \{\epsilon_0\}$ is strictly increasing on $[0,1]$. Then*

$$D_\beta(fp, fq) \leq \beta(p,q), \quad \textit{for every } \ p,q \in S.$$

If $(S,\mathcal{F},t)$ is a Menger space such that $t \geq t_m$ then $D_\beta(fp,fq) < \beta(p,q)$ for every $p \neq q$.

Proof. Let $r > 0$ be such that $0 < r < \frac{1-k}{k}\beta(p,q)$ where k is the contraction constant and $\beta(p,q) > 0$. Then $\beta(p,q) > k[\beta(p,q)+r]$ and since $\tilde{F}_{fp,fq}$ is strictly increasing we have that

$$\tilde{F}_{fp,fq}(\beta(p,q)) > \tilde{F}_{fp,fq}(k(\beta(p,q)+r)) \geq F_{p,q}(\beta(p,q)+r) > 1-\beta(p,q).$$

From the Definition of $\tilde{F}_{fp,fq}$ it follows that

$$\sup_{s<\beta(p,q)}\inf_{u\in fp}\sup_{v\in fq} F_{u,v}(s) > 1-\beta(p,q)$$

which implies

(1) $$\inf_{u\in fp}\sup_{v\in fq} F_{u,v}(\beta(p,q)) > 1-\beta(p,q)$$

and similarly

(2) $$\inf_{v\in fq}\sup_{u\in fp} F_{u,v}(\beta(p,q)) > 1-\beta(p,q).$$

From (1) it follows that for every $u \in fp$

$$\sup_{v \in fq} F_{u,v}(\beta(p,q)) > 1 - \beta(p,q)$$

and from (2) that for every $v \in fq$

$$\sup_{u \in fp} F_{u,v}(\beta(p,q)) > 1 - \beta(p,q).$$

This means that for every $u \in fp$ there exists $v(u) \in fq$ such that

$$F_{u,v(u)}(\beta(p,q)) > 1 - \beta(p,q) \tag{3}$$

and similarly for every $v \in fq$ there exists $u(v) \in fp$ such that

$$F_{u(v),v}(\beta(p,q)) > 1 - \beta(p,q). \tag{4}$$

Relation (3) implies that $\beta(u, v(u)) < \beta(p,q)$ and relation (4) implies that $\beta(u(v), v) < \beta(p,q)$. Hence

$$\inf_{v \in fq} \beta(u,v) < \beta(p,q), \inf_{u \in fp} \beta(u,v) < \beta(p,q)$$

which implies that

$$\sup_{u \in fp} \inf_{v \in fq} \beta(u,v) \leq \beta(p,q) \tag{5}$$

$$\sup_{v \in fq} \inf_{u \in fp} \beta(u,v) \leq \beta(p,q). \tag{6}$$

From (5) and (6) we have that

$$D_\beta(fp, fq) \leq \beta(p,q), \quad \text{for every } p, q \in S.$$

If $(S, \mathcal{F}, t)$ is a Menger space such that $t \geq t_m$ then $\beta(\cdot,\cdot)$ is a metric on S and $\psi(u) = \inf_{v \in fq} \beta(u,v)$ and $\varphi(v) = \inf_{u \in fp} \beta(u,v)$ are continuous function $\psi : fp \to \mathbb{R}^+$, $\varphi : fq \to \mathbb{R}^+$. Since fp and fq are compact there exist $u_0 \in fp$ and $v_0 \in fq$ such that

$$\psi(u_0) = \sup_{u \in fp} \psi(u), \ \varphi(v_0) = \sup_{v \in fq} \varphi(v).$$

Hence

$$\sup_{u \in fp} \inf_{v \in fq} \beta(u,v) = \inf_{v \in fq} \beta(u_0, v) < \beta(p,q)$$

and

$$\sup_{v\in fq}\inf_{u\in fp}\beta(u,v)=\inf_{u\in fp}\beta(u,v_0)<\beta(p,q).$$

This means that $D_\beta(fp,fq)<\beta(p,q)$ for $p\neq q$.

Proof of Theorem 1 . As in [8] for every $(p,q)\in S\times S,\ \ p\neq q$ there exists $\gamma_{p,q}\in(0,1)$ such that

$$D(fp,fq)<\gamma_{p,q}\beta(p,q).$$

Since the set $Ran(\mathcal{F})$ is finite there exists γ such that $\mathrm{Max}\{\gamma_{p,q};\ \ p,q\in S\}<\gamma<1$. Hence, for every $p,q\in S$

$$D_\beta(fp,fq)\leq\gamma\beta(p,q). \tag{7}$$

Remark 1. If $(S,\mathcal{F},t)$ is a Menger space such that $t\geq t_m$ and that (S,β) is a compact metric space using the Lemma we can obtain a fixed point result for $f:S\to Cl(S)$ (the family of nonempty closed subsets of S) which is a continuous $B-$contraction and $\tilde{\mathcal{F}}$ is such that every element of $Ran(\tilde{\mathcal{F}})\setminus\{\epsilon_0\}$ is strictly increasing.

Namely, the following result is well known [2]: *Let (S,d) be a compact metric space and $f:S\to Cl(S)$ a continuous mapping such that*

$$D(fp,fq)<d(p,q),\ \ p\neq q.$$

Then f has a fixed point.
Hence, we have the following result:

Proposition 1. *Let $(S,\mathcal{F},t)$ be a Menger space such that $t\geq t_m$ and (S,β) is a compact metric space, $f:S\to Cl(S)$ a continuous $B-$ contraction and every element of $Ran(\tilde{\mathcal{F}})\setminus\{\epsilon_0\}$ is strictly increasing. Then there exists $x\in S$ such that $x\in fx$.*

Remark 2. Using the well known Nadler's fixed point theorem we have the following result.

Proposition 2. *Suppose that all the conditions of Theorem 1 are satisfied and that (S,β) is a complete metric space. Then there exists $x\in S$ such that $x\in fx$.*

Remark 3. Suppose that (7) holds for every $p,q \in S$ and that $F_{p,q}(x) > 1-x$. This implies that $\beta(p,q) < x$ and (7) implies that $D_\beta(fp,fq) < \gamma x$. Using the definition of D_β we have that $\sup_{u\in fp} \inf_{v\in fq} \beta(u,v) < \gamma x$ and $\sup_{v\in fq} \inf_{u\in fp} \beta(u,v) < \gamma x$. Hence $\inf_{v\in fq} \beta(u,v) < \gamma x$, for every $u \in fp$ and $\inf_{u\in fp} \beta(u,v) < \gamma x$, for every $v \in fq$. Since fp and fq are compact for every $u \in fp$ there exists $v(u) \in fq$ such that

$$F_{u,v(u)}(\gamma x) > 1 - \gamma x$$

and similarly for every $v \in fq$ there exists $u(v) \in fp$ such that

$$F_{u(v),v}(\gamma x) > 1 - \gamma x.$$

From this we have that f is an H_2 – contraction.

Theorem 2. *Let* $(S,\mathcal{F})$ *be a probabilistic semimetric space and* $f : S \to nB(S)$ *an* H_1 – *contraction. Then* f *is an* H_2– *contraction and*

(8) $$D_\beta(fp,fq) \le k \cdot \beta(p,q) \quad \textit{for every } p,q \in S.$$

Proof. Let $\beta(p,q) < s$. We shall prove that

(9) $$D_\beta(fp,fq) \le ks$$

which implies (8).

From $\beta(p,q) < s$ it follows that $F_{p,q}(s) > 1 - s$. Since f is an H_1– contraction we have that

$$\tilde{F}_{fp,fq}(ks) > 1 - ks$$

and so

(10) $$\sup_{u<ks} \inf_{u\in fp} \sup_{v\in fq} F_{u,v}(u) > 1 - ks$$

(11) $$\sup_{u<ks} \inf_{v\in fq} \sup_{v\in fp} F_{u,v}(u) > 1 - ks.$$

Relations (10) and (11) implies that

(12) $$\inf_{u\in fp} \sup_{v\in fq} F_{u,v}(ks) > 1 - ks$$

$$\inf_{v \in fq} \sup_{u \in fp} F_{u,v}(ks) > 1 - ks \tag{13}$$

and so for every $u \in fp$ and $v \in fq$ we have that

$$\sup_{v \in fq} F_{u,v}(ks) > 1 - ks \tag{14}$$

$$\sup_{u \in fp} F_{u,v}(ks) > 1 - ks. \tag{15}$$

Hence, for every $u \in fp$ there exists $v(u) \in fq$ such that $F_{u,v(u)}(ks) > 1 - ks$ and for every $v \in fq$ there exists $u(v) \in fp$ such that $F_{v,u(v)}(ks) > 1 - ks$. Thus, $\beta(u, v(u)) < ks$ and $\beta(v, u(v)) < ks$ which implies that

$$D_\beta(fp, fq) \leq ks.$$

Corollary 1. *If $(S, \mathcal{F}, t)$ is a complete Menger space such that $t \geq t_m$ and $f : S \to CB(S)$ (closed and bounded subsets of S) an H_1- contraction then there exists $x \in S$ such that $x \in fx$.*

Theorem 3. *Let $(S, \mathcal{F}, t)$ be a probabilistic bounded Menger space with continuous T-norm t and $f : S \to Com(S)$ a C - contraction. Then f is a k-set probabilistic contraction.*

Proof. The proof of this theorem is in fact given in a part of the proof of Theorem 1 from [4] but we shall give it here for the completeness.

In order to prove the inequality

$$\beta_{f(A)}(ks) \geq \beta_A(s), \quad \text{for every} \quad s > 0 \tag{16}$$

and every $A \subset S$ we shall prove that for every $\epsilon \in (0, s)$

$$\beta_A(s - \epsilon) \leq \beta_{f(A)}(ks) \tag{17}$$

which implies (16), since $\beta_A(\cdot)$ is a left continuous function. If $\beta_A(s) = 0$ then (16) follows and suppose that $\beta_A(s) > 0$. In order to prove (17) we shall prove the implication:

$$r < \beta_A(s - \epsilon) \Rightarrow r < \beta_{f(A)}(ks) \tag{18}$$

since (18) implies (17). From $r < \beta_A(s-\epsilon)$ it follows that there exists a finite set $A_f = \{x_1, x_2, ..., x_n\} \subset S$ such that

$$\inf_{z \in A} \max_{w \in A_f} F_{z,w}(s-\epsilon) > r$$

and so for every $z \in A$ there exists $w(z) \in A_f$ so that $F_{z,w(z)}(s-\epsilon) > r$. If $y \in fz \ (z \in A)$ then there exists $x \in f(w(z))$ so that

$$F_{y,x}(k(s-\epsilon)) \geq F_{z,w(z)}(s-\epsilon).$$

Let $\delta \in (0, r)$ and $\lambda(\delta) \in (0,1)$ be such that $(t(r,1) = r)$

$$1 \geq u > 1 - \lambda(\delta) \Rightarrow t(r,u) > r - \delta$$

and for every $i \in \{1, 2, ..., n\}$ let

$$fx_i \subset \cup_{j=1}^{n(i)} U_{z_j^i}(\frac{k\epsilon}{2}, \lambda(\delta))$$

where

$$U_v(\epsilon, \lambda) = \{p;\ p \in S, F_{p,v}(\epsilon) > 1 - \lambda\}.$$

It can be proved that

$$\tilde{F}_{f(A), \cup_{i=1}^{n} \cup_{j=1}^{n(i)} \{z_j^i\}}(ks) > r\delta \tag{19}$$

and (19) implies $\beta_{f(A)}(ks) \geq r - \delta$. Hence $\beta_{f(A)}(ks) \geq r$ since δ is an arbitrary element from $(0, r)$.

Remark 4. It is obvious that f is densifying on S with respect to β if S is complete .

References

[1] Bocsan, Gh., Constantin, Gh.: The Kuratowski function and some applications to the probabilistic metric spaces, Sem. Teor. Funct. Si. Mat. Apl., Timisoara RS Romania, No. 1, 1973.

[2] Czerwik, S.: Fixed point theorems and special solutions of functional equations, Universytet Slaski, Katowice, 1980.

[3] Hadžić, O.: Some theorems on the fixed points in probabilistic metric and random normed spaces, Bull. Unione Mat. Ital. (6), 1-B (1982), 381-391.

[4] Hadžić, O.: Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces, Fuzzy Sets and Systems 29 (1989), 115-125.

[5] Hicks, T.L.: Fixed point theory in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod. - Mat. Fak. Ser. Mat. 13 (1983), 63-72.

[6] Radu, V.: Some fixed point theorems in probabilistic metric spaces, Lectures Notes in Math. 1233, 125-133.

[7] Schweizer B., Sklar, A.: Probabilistic Metric Spaces, Elsevier North - Holland, 1983.

[8] Schweizer, B., Sherwood, H., Tardif, R.M.: Contractions on probabilistic metric spaces, Examples and counter examples, Stochastica XII-1 (1988), 5-17.

[9] Sehgal, V.M., Bharucha-Reid, A.T.: Fixed points of contraction mappings in probabilistic metric spaces, Math. Systems Theory 6(1972), 97-102.

[10] Tan, D.H.: On probabilistic condensing mappings, Rev. Roum. Math. Pures Appl. 26,10(1981), 1305-1317.

REZIME

O VIŠEZNAČNIM KONTRAKCIJAMA U VEROVATNOSNIM METRIČKIM PROSTORIMA

Neke osobine višeznačnih kontrakcija u verovatnosnim metričkim prostorima su ispitane.

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