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A COMMON FIXED POINT THEOREM FOR TWO SEQUENCES OF MAPPINGS IN CONVEX METRIC SPACES

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Abstract

In this paper we shall give a generalization of Theorem A from [3] in convex metric spaces.

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1. Introduction

In [1] Assad and Kirk have proved the following theorem: Let (X, d) be a complete and convex metric space, C a nonempty closed subset of X, T a contraction mapping of C into $CB(X)$. If $T(\partial C) \subseteq C$ then there exists $u \in C$ such that $u \in Tu$.

In this theorem the convexity of X means that for each $x, y \in X$ with $x \neq y$, there exists $z \in X$, $x \neq z$, $y \neq z$, such that $d(x, z) + d(z, y) = d(x, y)$.

There are many fixed point theorems and common fixed point theorems in convex metric spaces for singlevalued and multivalued mappings and family of mappings.

T.Taniguchi generalized in [3] Theorem 1 from [2] . In this paper we shall prove a generalization of Theorem A from [3] in the case of a convex metrics space

2. The common fixed point theorem

Theorem 1. *Let (X,d) be a complete, convex metric space, K a nonempty, closed subset of X, $B_i : X \to X$ $(i \in \mathbf{N})$ and $A_i : K \to X$ continuous mappings so that $\partial K \subseteq B_i(K) \subseteq K$, $A_i K \cap K \subseteq B_{i+1} K$ and*

$$B_i x \in \partial K \Rightarrow A_i x \in K \quad \text{for every} \quad i \in \mathbf{N}.$$

Suppose that the following conditions are satisfied for all $m, n \in \mathbf{N}$ and all $x, y \in K$:

a) there exists a constant $k < 1$, such that

$$d(A_{2n-1}x, A_{2n}y) \leq kd(B_{2n-1}x, B_{2n}y)$$

$$d(A_{2n}x, A_{2m+1}y) \leq kd(B_{2n}x, B_{2m+1}y), \quad \text{for all} \quad m \geq n \geq 1.$$

b)

$$A_{2n}B_{2m}x = B_{2m}A_{2n}x \quad \text{and} \quad A_{2n-1}B_{2m-1}x = B_{2m-1}A_{2n-1}x$$

c)

$$B_{2n}B_{2m}x = B_{2m}B_{2n}x \quad \text{and} \quad B_{2m-1}B_{2n-1}x = B_{2n-1}B_{2m-1}x,.$$

Then there exists a unique common fixed point for two sequences $\{A_n\}_{n \in \mathbf{N}}$ and $\{B_n\}_{n \in \mathbf{N}}$.

Proof. Let $p \in \partial K$ and $p_0 \in K$ such that $p = B_1 p_0$. The existence of such an element follows from the condition $\partial K \subseteq B_1 K$. Since $B_1 p_0 \in \partial K$ it follows that $A_1 p_0 \in K$ and so from the condition $A_1 p_0 \in A_1 K \cap K \subseteq B_2 K$ it follows the existence of an element $p_1 \in K$ such that $B_2 p_1 = A_1 p_0$. Let $p_1' = A_1 p_0$ and $p_2' = A_2 p_1$. If $p_2' \in K$ it follows that $A_2 p_1 \in A_2 K \cap K \subseteq B_3 K$ and so the exists $p_2 \in K$ such that $B_3 p_2 = A_2 p_1$. If $p_2' \notin K$ then there exists $p_2 \in K$ such that $B_3 p_2 \in \partial K$ and

$$d(B_2 p_1, B_3 p_2) + d(B_3 p_2, A_2 p_1) = d(B_2 p_1, A_2 p_1).$$

If we proceed in this way we obtain two sequences $\{p_n\}_{n\in\mathbf{N}}$ and $\{p'_n\}_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$, $p_n \in K$, $p'_{n+1} = A_{n+1}p_n$ and the following implications hold for every $n \in \mathbf{N}$:

(i) $p'_{2n} \in K \Rightarrow p'_{2n} = B_{2n+1}p_{2n}$

$$p'_{2n} \notin K \Rightarrow B_{2n+1}p_{2n} \in \partial K \quad \text{and}$$

$$d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) + d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) = \\ = d(B_{2n}p_{2n-1}, A_{2n}p_{2n-1}).$$

(ii) $p'_{2n+1} \in K \Rightarrow p'_{2n+1} = B_{2n+2}p_{2n+1}$

$$p'_{2n+1} \notin K \Rightarrow B_{2n+2}p_{2n+1} \in \partial K \quad \text{and}$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) + d(B_{2n+2}p_{2n+1}, A_{2n+1}p_{2n}) = \\ = d(B_{2n+1}p_{2n}, A_{2n+1}p_{2n}).$$

We shall prove that there exists $z \in K$ such that $z = \lim_{n\to\infty} B_n p_{n-1}$. Let

$$P_0 = \{p_{2n};\ p'_{2n} = B_{2n+1}p_{2n}\},$$

$$P_1 = \{p_{2n};\ p'_{2n} \neq B_{2n+1}p_{2n}\},$$

$$Q_0 = \{p_{2n+1};\ p'_{2n+1} = B_{2n+2}p_{2n+1}\},$$

$$Q_1 = \{p_{2n+1};\ p'_{2n+1} \neq B_{2n+2}p_{2n+1}\}.$$

If $p_{2n} \in P_1$ then $B_{2n+1}p_{2n} \in \partial K$ which implies that $A_{2n+1}p_{2n} = p'_{2n+1} \in K$. From this it follows that $p'_{2n+1} = B_{2n+2}p_{2n+1}$ and so $p_{2n+1} \in Q_0$. It is easy to see that we have the following posibilities:

$$(p_{2n}, p_{2n+1}) \in P_0 \times Q_0;\ (p_{2n}, p_{2n+1}) \in P_0 \times Q_1;$$

$$(p_{2n}, p_{2n+1}) \in P_1 \times Q_0;$$

$$a) \qquad (p_{2n}, p_{2n+1}) \in P_0 \times Q_0$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) = d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq \\ \leq qd(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}).$$

$$b) \qquad (p_{2n}, p_{2n+1}) \in P_0 \times Q_1$$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) = d(B_{2n+1}p_{2n}, A_{2n+1}p_{2n}) -$$

$$d(B_{2n+2}p_{2n+1}, A_{2n+1}p_{2n}) \leq d(B_{2n+1}p_{2n}, A_{2n+1}p_{2n})$$

$$= d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq qd(B_{2n}p_{2n-1}, B_{2n+1}p_{2n})$$

c) $(p_{2n}, p_{2n+1}) \in P_1 \times Q_0$

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) \leq d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) +$$

$$+ d(A_{2n}p_{2n-1}, B_{2n+2}p_{2n+1}) =$$

$$= d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) + d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$

$$\leq d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) + qd(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$

$$\leq d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) + d(B_{2n+1}p_{2n}, A_{2n}p_{2n-1}) +$$

$$= d(B_{2n}p_{2n-1}, A_{2n}p_{2n-1}).$$

Since $p_{2n} \in P_1$ implies that $p_{2n-1} \in Q_0$ it follows that $B_{2n}p_{2n-1} = A_{2n-1}p_{2n-2}$ and so

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) \leq d(B_{2n}p_{2n-1}, A_{2n}p_{2n-1}) =$$

$$= d(A_{2n-1}p_{2n-2}, A_{2n}p_{2n-1}) \leq qd(B_{2n-1}p_{2n-2}, B_{2n}p_{2n-1}).$$

Similarly we can prove the following implications:

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_0 \Rightarrow d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$

$$\leq qd(B_{2n-1}p_{2n-2}, B_{2n}p_{2n-1});$$

$$(p_{2n-1}, p_{2n}) \in Q_1 \times P_0 \Rightarrow d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$

$$\leq qd(B_{2n-1}p_{2n-2}, B_{2n-2}p_{2n-3});$$

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_1 \Rightarrow d(B_{2n}p_{2n-1}, B_{2n+1}p_{2n}) \leq$$

$$\leq qd(B_{2n-1}p_{2n-2}, B_{2n}p_{2n-1}).$$

It is easy to prove that

$$d(B_{2n+1}p_{2n}, B_{2n+2}p_{2n+1}) \leq q^{n-1} \cdot r$$

$$d(B_{2n+2}p_{2n+1}, B_{2n+3}p_{2n+2}) \leq q^{n} \cdot r$$

where $r = \max\{d(B_3p_2, B_4p_3), \ d(B_3p_2, B_2p_1)\}$, which imply the existence of $z \in K$ such that

$$z = \lim_{n\to\infty} B_np_{n-1}.$$

There exists at least one sequence $\{B_{2n_k+1}p_{2n_k}\}_{k\in N}$ or $\{B_{2m_k+2}p_{2m_k+1}\}_{k\in N}$ such that $p_{2n_k} \in P_0(k \in \mathbf{N})$ or $p_{2m_k+1} \in Q_0(k \in \mathbf{N})$.

Suppose that there exists $\{n_k\}$ such that

$$B_{2n_k+1}p_{2n_k} = A_{2n_k}p_{2n_k-1}, \quad \text{for every} \quad k \in \mathbf{N}.$$

We shall prove that

(1) $$\lim_{k\to\infty} A_{2n_k+1}p_{2n_k} = z.$$

Relation (1) follows from

$$d(A_{2n_k+1}p_{2n_k}, B_{2n_k+1}p_{2n_k}) = d(A_{2n_k+1}p_{2n_k}, A_{2n_k}p_{2n_k-1})$$
$$\le qd(B_{2n_k+1}p_{2n_k}, B_{2n_k}p_{2n_k-1})$$

since

$$\lim_{k\to\infty} d(A_{2n_k+1}p_{2n_k}, B_{2n_k+1}p_{2n_k})$$
$$\le q \cdot \lim_{k\to\infty} d(B_{2n_k+1}p_{2n_k}, B_{2n_k}p_{2n_k-1}) = 0$$

and

$$\lim_{k\to\infty} B_n p_{n-1} = z.$$

Further, for every $m \in \mathbf{N}$

(2) $$B_m z = B_{m+1} z.$$

In order to prove (2) we shall prove that $B_{2m}z = B_{2m+1}z$, and $B_{2m}z = B_{2m-1}z$ for every $m \in \mathbf{N}$.

Since

$$z = \lim_{k\to\infty} A_{2n_k}p_{2n_k-1} = \lim_{k\to\infty} A_{2n_k+1}p_{2n_k}$$

it follows from the continuity of B_n that

$$d(B_{2m}z, B_{2m+1}z) = d(B_{2m}(\lim_{k\to\infty} A_{2n_k}p_{2n_k-1}), B_{2m+1}(\lim_{k\to\infty} A_{2n_k+1}p_{2n_k})) =$$
$$= \lim_{k\to\infty} d(B_{2m}A_{2n_k}p_{2n_k-1} B_{2m+1}A_{2n_k+1}p_{2n_k})$$
$$= \lim_{k\to\infty} d(A_{2n_k}B_{2m}p_{2n_k-1}, A_{2n_k+1}B_{2m+1}p_{2n_k}) \le$$
$$\le q \lim_{k\to\infty} d(B_{2n_k}B_{2m}p_{2n_k-1}, B_{2n_k+1}B_{2m+1}p_{2n_k})$$
$$= q \lim_{k\to\infty} d(B_{2m}B_{2n_k}p_{2n_k-1}, B_{2m+1}B_{2n_k+1}p_{2n_k}) =$$

$$= qd(B_{2m}z, B_{2m+1}z).$$

Since $q < 1$ it follows that $B_{2m}z = B_{2m+1}z$, for every $m \in \mathbf{N}$. We shall prove that $B_{2m}z = B_{2m-1}z$, for every $m \in \mathbf{N}$. Since $z = \lim_{k\to\infty} A_{2n_k}p_{2n_k-1} = \lim_{k\to\infty} A_{2n_k+1}p_{2n_k}$ it follows that

$$d(B_{2m}z, B_{2m-1}z) = d(B_{2m}(\lim_{k\to\infty} A_{2n_k}p_{2n_k-1}), B_{2m-1}(\lim_{k\to\infty} A_{2n_k+1}p_{2n_k}))$$

$$= \lim_{k\to\infty} d(B_{2m}A_{2n_k}p_{2n_k-1}, B_{2m-1}A_{2n_k+1}p_{2n_k}) =$$

$$= \lim_{k\to\infty} d(A_{2n_k}B_{2m}p_{2n_k-1}, A_{2n_k+1}B_{2m-1}p_{2n_k})$$

$$\leq q \lim_{k\to\infty} d(B_{2n_k}B_{2m}p_{2n_k-1}, B_{2n_k+1}B_{2m-1}p_{2n_k})$$

$$= q \lim_{k\to\infty} d(B_{2m}B_{2n_k}p_{2n_k-1}, B_{2m-1}B_{2n_k+1}p_{2n_k})$$

$$= qd(B_{2m}z, B_{2m-1}z)$$

and so $B_{2m}z = B_{2m-1}z$. Further, we have that $A_{2n}z = B_{2n+1}z$, $n \in \mathbf{N}$. Indeed for $n_k \neq n$ we have that

$$d(B_{2n+1}A_{2n_k+1}p_{2n_k}, A_{2n}z) = d(A_{2n_k+1}B_{2n+1}p_{2n_k}, A_{2n}z)$$

$$\leq qd(B_{2n_k+1}B_{2n+1}p_{2n_k}, B_{2n}z) = qd(B_{2n+1}B_{2n_k+1}p_{2n_k}, B_{2n}z)$$

and so

$$\lim_{k\to\infty} d(B_{2n+1}A_{2n_k+1}p_{2n_k}, A_{2n}z) \leq$$

$$\leq q \lim_{k\to\infty} d(B_{2n+1}B_{2n_k+1}p_{2n_k}, B_{2n}z).$$

Hence

$$d(B_{2n+1}z, A_{2n}z) \leq qd(B_{2n+1}z, B_{2n}z)$$

and since we can prove easily that $B_{2n+1}z = B_{2n}z$ we have that $B_{2n+1}z = A_{2n}z$, $n \in \mathbf{N}$. From the inequality

$$d(A_{2n-1}z, A_{2n}z) \leq qd(B_{2n-1}z, B_{2n}z)$$

we conclude that $A_{2n-1}z = A_{2n}z$, for every $n \in \mathbf{N}$.

Further, since $B_j : K \to K$ it follows that $A_jz \in K$, $j \in \mathbf{N}$ and

$$d(A_{2n-1}z, A_{2n}A_{2n}z) \leq qd(B_{2n-1}z, B_{2n}(A_{2n}z))$$

$$= qd(A_{2n-1}z, A_{2n}(B_{2n}z))$$

$$= qd(A_{2n-1}z, A_{2n}(A_{2n}z)).$$

This implies that $A_{2n-1}z = A_{2n}z = A_{2n}(A_{2n}z) = A_{2n}(B_{2n}z) = B_{2n}(A_{2n}z)$ and similarly

$$d(A_{2n}z, A_{2n+1}A_{2n+1}z) \leq qd(B_{2n}z, B_{2n+1}(A_{2n+1}z))$$
$$= qd(A_{2n}z, A_{2n+1}(B_{2n+1}z))$$
$$= qd(A_{2n}z, A_{2n+1}(A_{2n+1}z))$$

$A_{2n}z = A_{2n+1}(A_{2n+1}z) = A_{2n+1}(A_{2n}z) = A_{2n+1}(B_{2n+1}z) = B_{2n+1}(A_{2n+1}z) = B_{2n+1}A_{2n}z$.

Hence $u = A_{2n}z$ is a common fixed point for the families $\{A_n\}_{n\in\mathbf{N}}$ and $\{B_n\}_{n\in\mathbf{N}}$. The uniqueness of the common fixed point of the families $\{A_n\}_{n\in\mathbf{N}}$ and $\{B_n\}_{n\in\mathbf{N}}$ follows immediately since from $u = A_n u = B_n u$ and $w = A_n w = B_n w$ for every $n \in \mathbf{N}$ we have:

$$d(u, w) = d(A_{2n-1}u, A_{2n}w) \leq kd(B_{2n-1}u, B_{2n}w) \leq$$
$$\leq kd(u, w)$$

which implies $u = w$.

References

[1] Assad, N.A., Kirk, W.A.: Fixed point theorems of set-valued mappings of contractive type, Pacific J. Math. Vol. 43, No. 3 (1972), 553-562.

[2] Hadžić, O.: Common fixed point theorems for family of mappings in complete meric spaces, Math. Japan, 29 (1984), 127-134.

[3] Taniguchi, T.: A common fixed point theorem for two sequences of self - mappings, Internat. J. Math. Math. Sci. Vol. No. 3 (1991), 417-420.

REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI ZA DVA NIZA PRESLIKAVANJA U KONVEKSNIM METRIČKIM PROSTORIMA

Dokazano je uopštenje teoreme iz [3] u konveksnim metričkim prostorima.

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