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ON THE BEHAVIOUR OF THE STIELTJES TRANSFORMATION AT THE ORIGIN

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Abstract

The behaviour of the Stieltjes transformation at the origin is investigated for a function f from L^1_{loc}, $supp\ f \subset [0,\infty)$ with the asymptotic expansion of the form

$$\sum_{i=p}^{\infty} a_i t^i,\ t \to 0.$$

This new result follows from the known one for the asymptotic expansion at ∞. Since the notion of the quasiasymptotic expansion of distributions is used, the result can be formulated for the distributional Stieltjes transformation of distributions.

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1. Notations and known results

By $\mathcal{S}$ and $\mathcal{S}'$ we denote the space of rapidly decreasing functions and tempered distributions, respectively. The space $J'(r)$, $r \in \mathbf{R} \setminus (-\mathbf{N})$ is defined in [2] as a subspace of $\mathcal{S}'_+ = \{f \in \mathcal{S}';\ supp\ f \subset [0,\infty)\}$ consisting of all f of the form

(1) $$f = F^{(m)} \text{ for some } m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},$$

$$F \in L^1_{loc},\ supp\ F \subset [0, \infty),$$

such that

(2) $$\int_0^\infty \left| \frac{F(t)}{(t+x)^{r+m+1}} \right| dt < \infty,\ x > 0.$$

$I'(r)$ is the subspace of $J'(r)$ consisting of all $f \in J'(r)$ for which (1) holds and instead of (2) there holds

(3) $|F(t)| < C(1+t)^{r+m-\varepsilon}$, $t > 0$, for some $C = C(F)$, $\varepsilon = \varepsilon(F) > 0$.

The distributional Stieltjes tansformation S_r of index r, $r \in \mathbf{R} \setminus (-\mathbf{N})$,

(4) $$(S_r f)(z) = (r+1)_m \int_0^\infty \frac{F(t)}{(t+z)^{r+m+1}} dt,\ z \in \mathbf{C} \setminus (-\infty, 0].$$

where $(a)_m = a(a+1)\dots(a+m-1)$, $m \in \mathbf{N}$, $(a)_0 = 1$.
It is easy to see that $S_r f$ is a holomorphic function of the complex variable z on the domain $\mathbf{C} \setminus (-\infty, 0]$.
By L we denote a slowly varying function at ∞ (0^+). For the properties of such a function we refer to [5] .
In our investigations of the distributional Stieltjes transformation, the notion of quasiasymptotic behaviour at ∞ (0^+) plays a fundamental role ([1]).

Recall, $f \in \mathcal{S}'_+$ has the quasiasymptotic behaviour at ∞ (0^+) with respect to $k^\alpha L(k)$ ($(1/k)^\alpha L(1/k)$) with the limit $g \in \mathcal{S}'_+$ if

$$\lim_{k \to \infty} < \frac{f(kt)}{k^\alpha L(k)}, \varphi(t) > = < g(t), \varphi(t) >,\ \varphi \in \mathcal{S}$$

(5)

$$(\lim_{k \to \infty} < \frac{f(t/k)}{(1/k)^\alpha L(1/k)}, \varphi(t) > = < g(t), \varphi(t) >,\ \varphi \in \mathcal{S}).$$

It is well known that g in (5) must be of the form $g = C f_{\alpha+1}$, where

$$f_{\alpha+1}(t) = \begin{cases} \frac{H(t) t^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1 \\ \\ f^{(n)}_{\alpha+n+1}(t), & \alpha \leq -1 \text{ for some } n \in \mathbf{N} \text{ with } n + \alpha > -1 \end{cases} \qquad (t \in \mathbf{R})$$

where H is Heaviside' s function.
We shall need the following relation from [4].

Let $f \in J'(r)$ have the quasiasymptotic behaviour at 0^+ with respect to $(1/k)^\alpha L(1/k)$. Then,

$$(z/k)^{r+m+1}(S_r f)(z/k) = (r+1)_m (S_{r+m}\Phi)(k/z), \tag{6}$$

where $\Phi(t) = t^{r+m-1}F(1/t)$ for $t > 0$ and $\Phi(t) = 0$ for $t \le 0$. Obviously,

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^{r-\alpha-1}L_1(t)} = \frac{C}{\Gamma(\alpha+m+1)}$$

($L_1(k) = L(1/k)$ is slowly varying at ∞) and Φ is locally integrable on $\mathbf{R}$.

We also need the following proposition from [3]

Proposition 1. *Let $f \in L^1_{loc}(0,\infty)$ and let*

$$f(t) \sim \sum_{i=1}^{\infty} \frac{a_i}{t^i},\ t \to \infty.$$

Then for $r \in \mathbf{N}_0$ and $n \ge 2$ we have the asymptotic expansion at infinity

$$(S_r f)(k) \sim \frac{1}{k^{r+1}}(a_1 B_1 + T_1 + a_n m_0) + \sum_{m=1}^{n-1} \frac{(-1)^m a_m (r+1)_{m-1} \ln k}{(m-1)! k^{r+m}} +$$

$$+ \sum_{l=2}^{n-2} \frac{(-1)^l}{k^{r+l}} \left[\Gamma(r+l) \sum_{i=1}^{l-1} \frac{a_i (r+1)_i}{(l-i)\Gamma(l+1-i)\Gamma(r+i)(i-1)!} - \right.$$

$$\left. - \sum_{j=0}^{l-1} \frac{(r+1)_j T_{j+1}}{\Gamma(l-j)\Gamma(r+l+j)} + a_l \frac{(r+1)_l B_l^*}{(l-1)!} - m_{l-1}(r+1)_{l-1} \right] +$$

$$+ \frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}} \left[\sum_{i=1}^{n-2} \frac{a_i (r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i+1)!} - \right.$$

$$\left. - \sum_{j=0}^{n-3} \frac{(r+1)_j T_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n-1}(r+1)_{n-1} B_{n-1}^*}{(n-1)!} - m_{n-1}(r+1)_{n-2} \right] +$$

$$+ \frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1} \ln k / k^{r+n} + o(\ln k / k^{r+n}),$$

where

$$\left\{ \begin{array}{rl} B_s = & \frac{(-1)^{r+s-1}}{\Gamma(r+s)} (1/z \ln z/(1+z))^{(r+s-1)}|_{z=1} + \\ & + \int_1^\infty \ln u/(1+u)^{r+s-1} du \\ B_s^* = & 1/(r+s) B_s, \end{array} \right. \qquad s \in \mathbf{N} \tag{7}$$

$$T_1 = a_2 + \frac{a_3}{2} + \cdots + \frac{a_{n-1}}{(n-2)},$$

$$T_2 = \frac{a_3}{2!} + \frac{a_4}{3\cdot 2} + \cdots + \frac{a_{n-1}}{(n-2)(n-3)},$$

$$T_3 = \frac{a_4}{3!} + \frac{a_5}{4\cdot 3\cdot 2} + \cdots + \frac{a_{n-1}(n-5)!}{(n-2)!},$$

$$\vdots$$

$$T_{n-2} = \frac{a_{n-1}}{(n-2)!}$$

and

$$m_i = \int_0^\infty t^i \frac{f(t) - \sum_{j=1}^{n-1} a_j H(t-1)/t^j}{i!} dt, \quad i = 1, \dots, n-2.$$

2. Application

We shall investigate the behaviour of the Stieltjes transformation at the orgin for a function f, $supp\ f \subset [0,\infty)$ and $f \in L^1_{loc}(A,\infty)$, $A > 0$, with the ordinary asymptotic expanion of the form

$$\sum_{i=p}^{\infty} a_i t^i, \quad t \to 0$$

using (6) and Proposition 1.

Let $m = 0$, $z = 1$, then from (6) we obtain

$$(1/k)^{r+1}(S_r f)(1/k) = (S_r \Phi)(k), \tag{8}$$

where

$$\Phi(t) = t^{r-1} f(1/t).$$

Let $r \in \mathbf{N}_0$, $supp\ f \subset [0,\infty)$, $f \in L^1_{loc}(A,\infty)$ for some $A > 0$ and

$$f(t) \sim \sum_{i=p}^{\infty} a_i t^i,\ t \to 0,$$

then

$$\Phi(t) = t^{r-1} f(1/t) \sim \sum_{i=p}^{\infty} \frac{a_i}{t^{i+1-p}} = \sum_{j=p+1-r}^{\infty} \frac{a_{j+r-1}}{t^j},\ t \to \infty$$

and $\Phi \in L^1_{loc}(0,1/A)$. If $p \geq r$ then the integral of Φ over $(1/A,\infty)$ is finite. From Proposition 1 we have:

Proposition 2. *Let supp* $f \subset [0,\infty)$, $f \in L^1_{loc}(A,\infty)$ *for some* $A > 0$ *and*

$$f(t) \sim \sum_{i=p}^{\infty} a_i t^i,\ t \to 0.$$

Then for $n \geq p+2-r$,

$$(S_r f)(s) \sim \tilde{T}_1 + a_{n+r-1}\tilde{m}_0 - \tag{9}$$

$$-\sum_{m=p-r+1}^{n-1} \frac{(-1)^m a_{m+r-1}(r+1)_{m-1}}{(m-1)!} s^{m-1}\ln s +$$

$$+\sum_{l=2}^{n-2}(-1)^l s^{l-1}\left[\Gamma(r+l)\sum_{i=p-r+1}^{l-1}\frac{a_{i+r-1}(r+1)_i}{(l-i)!\Gamma(l+1-i)\Gamma(r+i)(i-1)!} -\right.$$

$$\left.-\sum_{j=0}^{l-1}\frac{(r+1)_j\tilde{T}_{j+1}}{\Gamma(l-j)\Gamma(r+1+j)} + a_{l+r-1}(r+1)_l\frac{B_l^*}{(l-1)!} - \tilde{m}_{l-1}(r+1)_{l-1}\right] +$$

$$+(-1)^{n-1}\Gamma(r+n-1)s^{n+l-1}\left[\sum_{i=p-r+1}^{n-2}\frac{a_{i+r-1}(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+1)(i+1)!} -\right.$$

$$\left.-\sum_{j=0}^{n-3}\frac{(r+1)_j\tilde{T}_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n+r-2}(r+1)_{n-1}}{(n-1)!} - \tilde{m}_{n-1}(r+1)_{n-2}\right] -$$

$$-\frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1}s^{n-1}\ln s + o(s^{n-1}\ln s),\ s \to 0$$

where

$$\tilde{T}_1 = \frac{a_p}{p-r} + \cdots + \frac{a_{n+r-2}}{(n-2)},$$

$$\tilde{T}_2 = \frac{a_{p+1}}{(p-r+1)(p-r)} + \cdots + \frac{a_{n+r-2}}{(n-2)(n-3)},$$

$$\tilde{T}_3 = \frac{a_{p+2}}{(p+r-2)(p-r+1)(p-r)} + \cdots + \frac{a_{n+r-2}(n-5)!}{(n-2)!}, \tag{10}$$

$$\vdots$$

$$\tilde{T}_{n-2} = \frac{a_{n+r-2}}{(n-2)!}$$

(11) $$\tilde{m}_i = \int_0^\infty \frac{t^i}{i!}\left[t^{r-1}f(1/t) - \sum_{j=p-r+1}^{n-1} a_{j+r-1}H(t-1)/t^j\right]dt,$$

$i = 0, 1, \ldots, n-2$ and $B_s^*, s \in \mathbf{N}$, is introduced by (7).

Proof. From (8) it follows that

$$(1/k)^{r+1}(S_r f)(1/k) = (S_r\Phi)(k) \sim$$

$$\sim (1/k)^{r+1}(\tilde{T}_1 + a_{n+r-1}\tilde{m}_0) + \sum_{m=p-r+1}^{n-1}\frac{(-1)^m a_{m+r-1}(r+1)_{m-1}\ln k}{(m-1)!k^{r+m}} +$$

$$+\sum_{l=2}^{n-2}\frac{(-1)^l}{k^{r+l}}\left[\Gamma(r+l)\sum_{i=p-r+1}^{l-1}\frac{a_{i+r-1}(r+1)_i}{(l-i)!\Gamma(l+1-i)\Gamma(r+i)(i-1)!} -\right.$$

$$\left. -\sum_{j=0}^{l-1}\frac{(r+1)_j\tilde{T}_{j+1}}{\Gamma(l-j)\Gamma(r+1+j)} + a_{l+r-1}\frac{(r+1)_l B_l^*}{(l-1)!} - \tilde{m}_{l-1}(r+1)_{l-1}\right] +$$

$$+\frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n+l}}\left[\sum_{i=p-r+1}^{n-2}\frac{a_{i+r-1}(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+1)(i+1)!} -\right.$$

$$\left. -\sum_{j=0}^{n-3}\frac{(r+1)_j\tilde{T}_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)} - \frac{a_{n+r-2}(r+1)_{n-1}}{(n-1)!} - \tilde{m}_{n-1}(r+1)_{n-2}\right] +$$

$$+\frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1}\frac{\ln k}{k^{r+n}} + o\left(\frac{\ln k}{k^{r+n}}\right),$$

where $\tilde{T}_i$, $i = 0, 1, \ldots, n-1$, is given by (10) and $\tilde{m}_i$ is given by (11). If we put $1/k = s$, then we obtain the Proposition 2.

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REZIME

PONAŠANJE STIELTJESOVE TRANSFORMACIJE U NULI

Dokazano je tvrdjenje o ponašanju Stieltjesove transformacije u nuli, funkcije f sa osobinom *supp* $f \subset [0,\infty)$, $f \in L^1_{loc}(A,\infty)$ za neko $A > 0$ ako f ima asimptotski razvoj oblika

$$\sum_{i=p}^{\infty} a_i t^i, \ t \to 0.$$

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