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ON SUMMABLE BOUNDEDNESS IN CONVERGENCE LINEAR SPACES

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Abstract

Some properties of K-bounded sets and N-bounded sets in sequential convergence linear spaces are proved and some criterions for N-boundedness are obtained. Examples of K-konvergent, (N-konvergent) and K-bounded (N-bounded) sets are presented.

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1. Introduction

A kind of summable property of convergences in topological vector spaces was orginally introduced by S.Mazur and W.Orlicz in 1953 [12]. Recently

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this property was rediscovered by members of the Katowice Branch of the Mathematical Institute of Polish Academy of Scince. Many authors developed the theory of K-convergent spaces (K-Katowice) for example in [1,2,3,6,9,13-17,20]. The notion of K-convergence has proven to be quite useful and interesting in studying various topics in functional analysis. This is particularly true in the case of the uniform boundedness principle, where the notion of K-boundedness has led to versions of the uniform boundedness principle which are valid in the absence of any completeness assumptions ([1,2,3,20]).
Recently it has been applied in the theory of Adjoint Theorem ([15],[16],[17]), which has an application in the theory of Closed Graph Theorems.

Another kind of summable property of convergences, the so-called N-property (N = Novosibirsk) has been introduced in [19]. In this paper we shall introduce the notion of N-boundedness. We shall also prove some properties of K-bounded sets in convergences linear spaces. We shall obtain some criteria for N-boundedness. We shall present an example which illustrates the relationship between the notion of K-convergent (N-convergent) sequences and K-boundedness (N-boundeness).

2. Convergences, K-sequences and N-sequences

The convergence of sequences of elements in topological spaces is well known. It can be seen as a function which to some sequences assigns elements called limits. It can be also seen as a set of pairs $((x_n), x)$ such that x is a limit of sequence (x_n). In analysis we also deal with cases when convergence of sequences is introduced directly without using topology. Examples for this are pointwise convergence, convergence almost everywhere and others.

In this paper by a convergence in a set X we mean a set G of pairs $((x_n), x)$, where x_n for $n \in N$ and x are elements of X. The meaning of expression $((x_n), x) \in G$ is that the sequence (x_n) converges to x, or equivalently, x is a limit of (x_n). Instead of $((x_n), x) \in G$ we shall equivalently write $x_n \xrightarrow{G} x$, or simply, $x_n \to x$ ([4],[14]). To be able to prove properties of a convergence, or theorems concerning notions defined by the convergence we have to postulate some properties as axioms of the convergence. In this paper we are interested in relations between properties of a convergence and properties of K-convergent, N-convergent sequences and boundedness.

In the sequel we shall be concerned with convergences in Abelian groups and linear spases. We start with recalling the notion of K-sequences and N-sequences. Assume that X is an Abelian group endowed with a convergence G and (x_n) is a sequence of elements x_n in X. It will be convinient to say that a sequence (x_n) is series convergent to x and write

$$\sum_{n=1}^{\infty} x_n = x \quad \text{if} \quad \sum_{k=1}^{n} x_k \to x.$$

It may happen that there is no series convergent sequence when no property is required from G. However, we have the following evident proposition.

Proposition 1.

(a) *If G satisfies the condition*
(S') $x_n \to 0$ *whenever* $x_n = 0$ *for* $n \in N$, *then zero sequences are series convergent to zero.*

(b) *If G satisfies the conditions*
(F) $x_n \to x$ *implies* $x_{m_n} \to x$

and
(L_1) $x_n \to x$ *and* $y_n \to y$ *implies* $x_n - y_n \to x - y$,

then series convergent sequences converge to zero.

It is well known that the completness of metric spaces is of great importance. The procedure of completion makes sense in the case of metric spaces but this is not the case when we deal with convergence spaces. It apears that in many cases the completeness property may be successfully replaced by a summable property of sequences ([1], ,[2]). This was first noticed by Orlicz and Mazur in their paper [12] and independently it was rediscovered and studied at the seminar at Katowice ([1],[2],[5],[9]) conducted by J.Mikusinski and P.Antosik. To be more exact we adopt the following definition.

Definition 1. *A sequence (x_n) of elements in an Abelian group X endowed with a convergence is said to be a K-convergent sequence or shorter K-sequence if each subsequence of (x_n) has a series convergent subsequence.*

If (S') holds, then zero sequences are K-sequences. Sequences converging to zero may not be a K-sequence. For instance, if X is the group of rational numbers with the usual convergence, then the sequence (10^{-10^n}) is not a K-sequence.

Proposition 2. *If a convergence in X satisfies (F), (L_1) and the condition*
(U) $x_n \not\to 0$ implies that there exists a subsequence (x_{m_n}) of (x_n) whose no subsequence converges to zero,
then K-sequences converge to zero,

If $x_n \to x$ implies $(x_n - x)$ is a K-sequence we say that X is a K-convergence group. In [9] is given an example of a K-normed linear space, but not complete. In [6] it is proved that K metric groups are of the second category. There are examples of K-topological groups but not of the second category (see [8]).

The more restrictive summable property of sequences has been introduced and studied by S.L.Sobolev in his monograph [19].

Definition 2. *A sequence (x_n) of elements in an Abelian group X endowed with a convergence is said to be an N-sequence, if each subsequence of (x_n) has a subsequence (y_n) which is subseries convergent, i. e. for each subsequence (z_n) of (y_n) the series*

$$\sum_{n=1}^{\infty} z_n$$

converges to an element in X.

Obviously, N-sequences are K-sequences. The converse is not true (see [9]). If $x_n \to x$ in X implies $(x_n - x)$ is an N-sequence, we say that X is an N-convergence group. Recently *Burzyk* [5] has constructed an N-normed space (a dense subspace of c_o) wich is not complete.

3. Boundedness, K-boundedness and N-boundedness

In Funcional Analysis we deal with theorems on uniform bounded ness of families of continuous and linear mappings on bounded subset of the domain of mappings. The theorems are valid under the assumption of completness or Baire category or barelledness type. From the results of paper [2] (see, Theorem 1, p.4), it folows that pointwise boundedfamilies of linear and continuous mappings from a convergence linear space to a topological vector space are uniformly bounded on so-colled K-bounded sets. Bearing in mind to above mentioned result, it is interesting to now the conditions under which bounded sets are K-bounded.

Throughout this section we assume that X is a linear space endowed with a convergence G. We assume that the field of scalars is equipped with the usual convergence. So, the field of scalars is an N-convergence field.

Definition 3. *A subset B of X is bounded, (K-bounded, N-bounded), if for each sequence (x_n) of elements x_n in B and for each sequence (α_n) of scalars α_n tending to zero the sequence $(\alpha_n x_n)$ is convergent to zero (K-sequence, N-sequence).*

Obviously, N-bounded subsets of X are K-bounded. The converse is not true even for normed spaces ([9]). In general, N-bounded sets may not be bounded. Take, for instance, the space of measurable function on [0,1] with the almost everywhere convergence.

This can be true even in the case of N-convergence space.

Example 1. Let R be the field of real numbers endowed with convergence G such that $x_n \xrightarrow{G} x$ if $\sum_{n=1}^{\infty} |x_n - x| < \infty$. Then (R, G) is an N-convergence space in which the sequence $(\frac{1}{n})$ is N-bounded but not bounded. The reason for that lies in the fact that $\frac{1}{n} \xrightarrow{G} 0$, but condidion (U) does not hold.

A K-bounded sequence can be not N-bounded even in a K-normed space.

Example 2. We shall take the spaces from the Corollary of Theorem 3 from [11]. Let e_n $(n \in N)$ be unit vectors from l^2. Let $Y = lin\{e_n, n \in N\}$. Then there are two K-subspaces X_1 and X_2 of l^2, such that $X_1 \cap X_2 = Y$ and for each linearly independent sequence (x_n) tending to zero in l^2 there are two subsequences (x_{1n}) and (x_{2n}) of (x_n), such that

$$\sum_{n=1}^{\infty} x_{1n} \in X_1 \text{ but } \sum_{n=1}^{\infty} x_{1n} \notin X_2 \text{ and}$$

$$\sum_{n=1}^{\infty} x_{2n} \notin X_1 \text{ but } \sum_{n=1}^{\infty} x_{2n} \in X_2$$

Then (e_n) is a K-bounded sequence in X_1, but it is not N-bounded. Namely, if we take a sequence (α_n) of scalars α_n $(n \in N)$tending zero, then the sequence $(\alpha_n\ e_n)$ is a K-sequence in X_1 and it is linearly independent and $\alpha_n\ e_n \rightarrow 0$. Then , each subsequence of $(\alpha_n\ e_n)$ has a subsequence whose sum belongs to X_2 and does not belong to X_1, i.e. (e_n) is not N-bounded.

If X is a FLUSH linear convergence space (S - stacionarity and H - Hausdorff property,see [4],[14]), then each K-sequence is convergent to zero. Hence, each K-bounded set is bounded but the converse is not always true. But, there are a lot of important spaces for which the converse is also true.
Example 3. A topological vector space (X, τ) is said to be an A-space, if every τ-bounded subset of X is τ-K-bounded ([10],[17]). There are many interesting A-spaces (see [10]). For example, if X is a B-space, then $(X, \sigma(X, X'))$ is an A-space which is not barrelled and not even infrabarrelled (Theorem 5. from [10]). There exist A-spaces which are not K-spaces. For example $(l^p, weak)$, $1 < p < \infty$.

Proposition 3.

(a) *If the convergence in X satisfies (S'), then the set $\{0\}$ is bounded, (K-bounded, N-bounded).*

(b) *If convergence in X satisfies (F), (L_1) and (U), then K-bounded sets are bounded.*

(c) *Subset of X are bounded (K-bounded, N-bounded) if their countable subsets are bounded, (K-bounded, N-bounded).*

(d) *Finite unions of K-bounded (N-bounded) sets are K-bounded (N-bounded).*

(e) *Products by scalaras of bounded (K-bounded, N-bounded) sets are (K-bounded, N-bounded).*

Under additional conditions imposed on the convergence in X we get further properties similar to properties of bounded subsets od topological vectors.

Proposition 4.

(a) *If G satisfies the condition (M_α) $\alpha_n \to 0$ implies $\alpha_n x \to 0$ for each $x \in X$,*
then finite subsets of X are K-bounded (N-bounded). If, additionally, (U) holds,then finite subsets of X are bounded.

(b) *Finite algebraic sums of N-bounded subsets of X are N-bounded, whenever G satisfies the condition*
(L_2) *$x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$.*

We note that (S) and (L_1) imply L_2, but (S) and (L_2) do not imply (L_1).

Proposition 4(b) is not valid for K bounded sets. This is due to the fact that Cartesian products of K-convergence groups may not be K-convergence groups. This can be shown in the following way .

Example 4. Let e_i for $i \in N$ be unit vectors in l^2 and let $E = span\{e_i : i \in N\}$. E is a dense linear subspace of l^2. Following the proof of Theorem 2 in [9], we find K-subspaces E^1 and E^2 of l^2 such that $E^1 \cap E^2 = E$. We note that $E^1 \times E^2$ is not a K-space. In other words, the Cartesian product of K-spaces may not be a K space. In fact, consider $x_n = \frac{1}{n}e_n$ for $n \in N$. Then, $x_n \in E$ for $n \in N$, $x_n \to 0$ and $(x_n, x_n) \to 0$. Suppose that there exists a summable subsequence (x_{m_n}, x_{m_n}), and $\sum_{n=1}^{\infty} (x_{n_m}, x_{n_m}) = (\sum_{n=1}^{\infty} x_{n_m}, \sum_{n=1}^{\infty} x_{n_m}) = (x, y)$. This implies that $x = y$, $x \in E^1$, $x \in E^2$. Consequently, $x \in E$, since $E^1 \cap E^2 = E$. On the other hand, $x = (\frac{1}{m_1}, \frac{1}{m_2}, ...) \notin E$. This contradiction shows that $E^1 \times E^2$ is not a K-space.

We note that the sequence $(x_n, 0)$, $(0, x_n)$ $(n \in N)$ are K-bounded in E^1 and E^2, respectively. But $(x_n, 0) + (0, x_n) = (x_n, x_n)$ and so (x_n, x_n) $(n \in N)$ is not K-bounded. This shows that the algebraic sums of K-bounded sets may not be K-bounded.

Proposition 5.

(a) *If X is a K-convergence group and Y is an N-convergence group, then the Cartesian product $X \times Y$ is a K-convergence group.*

(b) *If A is a K-bounded subsets of X and B is an N-bounded subsets of X, then $A + B$ is a K-bounded subsets of X.*

Example 5. It is interesting to note that the weak convergence in l^2 is not a K-convergence. The sequence (e_n) converges weakly to zero, but it is not K-convergent. At the same time, the sequence is weakly bounded and K-bounded. Moreover, from the results in [3] (see Proposition 1), it follows that weakly bounded subsets of l^2 are K-bounded.
In what follows we shall study the conditions under which bounded subsets of linear space X equipped with a convergence G are K-bounded. To do this we adopt the following definitions.

A sequence (x_n) of elements in X is said to be Cauchy, if for each increasing sequence (p_n) of positive integers $x_{p_{n+1}} - x_{p_n} \rightarrow 0$. Under conditions (L_1), convergent sequences are Cauchy. A subset A of X is complete, if each Cauchy sequence of elements in A converges to an element in A. Closed subsets of a complete topological group are complete.

A sequence (x_n) of elements in X is said to be convex convergent if for any sequence of finite systems of positive integers $p_{n_1}, \ldots, p_{nk_n}$, such that $min(p_{n_1}, \ldots, p_{nk_n}) \rightarrow \infty$ and for any sequences of finite systems of nonnegative scalars $\alpha_{p_{n1}}, \ldots, \alpha_{p_{nk_n}}$ such that $\alpha_{p_{n1}} + \ldots + \alpha_{p_{nk_n}} \leq 1$, the sequence of sums

$$\alpha_{p_{n1}} x_{p_{n1}} + \ldots + \alpha_{p_{nk_n}} x_{p_{nk_n}}$$

converges to zero.

Proposition 6. *Subsequence of convex cconvergent sequences are convex convergent.*

Convex convergent sequences converge to zero. Sequences converging to zero in a convex topological vector space are convex convergent.In general, sequences converging to zero in F-spaces are not convex convergent (see |2|, p.10). However, sequences converging to zero in F spaces have convex convergent subsequences.This observation leads us to the following definition. A convergence G in a linear spaces X is said to be subsequentially convex convergent if each sequence converging to zero in X has a convex convergent subsequence ([2], p.10). F-spaces are subsequentially convex convergent. We have the following theorem.
Theorem *If G is a complete subsequentially convex convergence in a linear space X, then bounded subsets of X are N-bounded.*

Proof. Assume that A is a bounded subset of linear space X endowed with a complete subsequentially convex convergence. Suppose that $x_n \in A$

for $n \in N$ and (α_n) is a sequence of scalars tending to zero. We should show that $(\alpha_n x_n)$ is an N-sequence. To this end we take a subsequence $(\beta_n y_n)$ of $(\alpha_n x_n)$ and, next, we take a subsequence $\gamma_n t_n$ of $(\beta_n y_n)$ such that

$$\sum_{n=1}^{\infty} \tau_n \leq 1$$

with

$$\tau_n = \sqrt{|\gamma_n|}$$

for $n \in N$. We note that $(\tau_n^{-1} \gamma_n t_n)$ is a subsequentially convex convergent sequence. We may assume that the sequence is convex convergent. Otherwise, we would take a convex convergent subsequence of the sequence. We assert that the series

$$(1) \qquad \sum_{n=1}^{\infty} \gamma_n t_n$$

is convergent. In fact, assume that

$$u_n = \tau_n^{-1} \gamma_n t_n$$

for $n \in N$. We have $\tau_n u_n = \gamma_n t_n$ for $n \in N$. We put

$$S_n = \sum_{k=1}^{n} \tau_k u_k$$

. Let (p_n) be an increasing sequence of positive integers. We have

$$S_{p_{n+1}} - S_{p_n} = \sum_{k=p_n}^{p_{n+1}} \tau_k u_k$$

for $n \in N$. Since (u_k) is a convex convergent sequence and

$$\sum_{k=p_n}^{p_{n+1}} \tau_k \leq 1,$$

we have

$$S_{p_{n+1}} - S_{p_n} \to 0.$$

This shows that (1) is a Cauchy series. Since X is complete, then sum of (1) is in X. Hence, by Proposition 6, each subseries of (1) is convergent, which

means that $(\alpha_n x_n)$ is an N-convergent sequence.
Consequently A is an N-bounded set.

Under the same assumptions as in the Theorem, the conclusion of Proposition 2 in [2] is that bounded subsets of X are K-bounded. So, our Theorem is a stronger result than Proposition 2 in [2].

It is well known that a continuous linear map between topological vector spaces is bounded, i.e. preserves bounded sets. It is easy to prove

Proposition 7. *A continuous linear map between convergence linear spaces preserves K-bounded (N-bounded) sets.*

Taking the identity map in the preceding proposition, we obtain
Corollary. If G_1 and G_2 are two convergences on a linear space X such that $G_2 \subset G_1$, then each subset of X which is bounded, K-bounded or N-bounded in G_2 is also bounded, K-bounded or N-bounded, respectively in G_1.

4. Convergence and Boundedness

We note that in case of the topological vector spaces, convergent sequences are bounded, more exactly, sets of members of convergent sequences are bounded. We assume that X and G are as in the section 3.

If G satisfies the condition

$$(M) \quad \alpha_n \to 0 \text{ and } x_n \to x \text{ implies } \alpha_n x_n \to 0,$$

then convergent sequences in X are bounded.

It is interesting to note that if G is induced by a group topology on X and the multiplication by scalar αx is separately continuous, then (M) holds (see [1]). In light of the definitions of K-boundedness and N-boundedness natural questions which suggest themselves are:

$$(Q_1) \text{Are } K-\text{convergent sequences } K-\text{bounded?}$$

$$(Q_2) \text{Are } N-\text{convergent sequences } N-\text{bounded?}$$

Of course, in a metrizable sequentially complete linear topologocal space, the answer to Q_1 and Q_2 is obviously: yes. We shall show that the answer are generally: no.

First, we present an example of K-convergent sequence which is not K-bounded (see [3]).

Example 6. Let l^∞ be the space of all the bounded real sequences. Let m_0 be the subspace of l^∞ which consist of those sequences (t_j) with a finite range. Pick an element $= (x_j)$ in l^1 such that $x_j \neq 0$ for each j (for example, $x_j = \frac{1}{j^2}$). Define a norm on m_0 induced by x, by means of the formula,

$$\|(t_j)\| = \|(t_j)\|_x = \sum_{j=1}^{\infty} |t_j x_j|.$$

Let e_j be the element of m_0 which has a 1 in the j^{th} coordinate and 0 elsewhere.

We first claim that the sequence (e_j) in $(m_0, \|\ \ \|_x)$ is N-convergent. This is immediate, since for any subsequence (e_{k_j}) if $y = (y_i) \in m_0$ is defined by $y_i = 1$ if $i = k_j$ for some j and $y_i = 0$ otherwise, we have

$$\|y - \sum_{j=1}^{n} e_{k_j}\| = \sum_{j=n+1}^{\infty} |x_{k_j}| \to 0.$$

The sequence (e_j) is not K-bounded in $(m_0, \|\ \|_x)$, since if (t_j) is any sequence of scalars which converges to 0 with $t_j \neq t_i$ for $i \neq j$, then no subseries of $\sum t_{k_j} e_{k_j}$ will converge to an element of m_0. (Any subseries $\sum t_{k_j} e_{k_j}$ will converge coordinatewise to an element of $l^\infty \setminus m_0$).

The method used in the construction of the example above can be generalized to yield a large number of examples of K-convergent sequences which are not K-bounded (see [3], p.15).

The above example 6 shows that even N-convergent sequences are not K-bounded. This gives negative answers to both questions.

K-bounded sets are also ipmortant in the construction of the locally convex topology which is used in the theory of the uniform Boundedness Principle ([2],[3],[20]) and in the theory of the Adjoint Theorem ([15],[16],[17]).

Example 7. Let X be a locally convex space and X', its dual. If we denote by $K(X, X')$, the topology of uniform convergence on $\sigma(X', X)$ - K-bounded subsets of X', then $K(X, X')$ is stronger than the Mackey topology $\tau(X, X')$ (Proposition 1 from [20]), but still have same bounded sets.

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REZIME

O SUMABILNOJ OGRANIČENOSTI U KONVERGENTNOM VEKTORSKOM PROSTORU

U radu su dokazane neke osobine K-ograničenih i N-ograničenih skupova u nizovno konvergentnom vektorskom prostoru. Dobijeni su i neki kriterijumi za N-ograničenost. Dat je primer K-konvergentnog (N-konvergentnog) niza koji nije K-ograničen (N-ograničen).

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