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SOME CLASSES OF DIHEDRAL n-QUASIGROUPS

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Abstract

An n-quasigroup (Q, f) is called dihedral iff $f(x_1, ..., x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = x_{\sigma(n+1)}$ for every permutation $\sigma \in D_{n+1}$, where D_{n+1} is the dihedral subgroup of the symmetric group S_{n+1} of degree $n + 1$. Dihedral n-quasigroups (D-n-quasigroups) represent a generalization of totally symmetric binary quasigroups. In the paper several classes of D-n-quasigroups are considered: (i,j)-associative D-n-quasigroups, D-n-groups, medial D-n-quasigroups, self-orthogonal D-n-quasigroups and D-n-quasigroups satisfying Menger identities. Their properties are described and some characterizations given.

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1. Introduction and Definitions

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty.

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An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \ldots, n\} = N_n$.

An n-quasigroup (Q, f) is called (i,j)-associative iff the following identity holds

(1) $$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

An n-quasigroup which is (i,j)-associative for all $i, j \in N_n$ is called an n-group.

An n-quasigroup (Q, f) is called m-associative iff for every $i, j \in N_n$ and every sequence x_1^{2n-1} of elements from Q which contains at most m different elements (1) holds.

An n-quasigroup (Q, f) is medial iff $f(y_1^n) = f(z_1^n)$, where $y_i = f(\{x_{ij}\}_{j=1}^n)$, $z_j = f(\{x_{ij}\}_{i=1}^n)$ for all $x_{ij} \in Q$, $i, j \in N_n$.

By S_n we denote the symmetric group of degree n, by A_n its alternating subgroup, and by D_n its dihedral subgroup.

If (Q, f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup f^σ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \Longleftrightarrow f(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of f. If $f = f^\sigma$, then σ is called an autoparastrophism of f. The set of all autoparastrophisms of f is a subgroup of S_{n+1} which is denoted by $\Pi(f)$.

An n-quasigroup $(Q.f)$ is called

a) totally symmetric (TS) if $f = f^\sigma$ for all $\sigma \in S_{n+1}$,

b) alternating symmetric (AS) iff $f = f^\sigma$ for all $\sigma \in A_{n+1}$ ([5]),

c) cyclic iff $f = f^\sigma$ for all $\sigma \in C$, where C is a subgroup of S_{n+1} generated by the cycle $(12 \ldots n+1)$ ([4]).

In [7] a class of n-quasigroups called dihedral n-quasigroups was defined and considered. An n-quasigroup (Q, f) is called dihedral iff $f = f^\sigma$ for all $\sigma \in D_{n+1}$. Such n-quasigroups represent a generalization of TS binary quasigroups, different from other generalizations such as TS-n-quasigroups or AS-n-quasigroups. In [6] a class of D-3-quasigroups which is equivalent to a new class of quadruple systems which lies between Steiner and Mendelsohn quadruple systems was cosidered.

An n-quasigroup (Q, f) is dihedral iff $f = f^{\phi} = f^{\psi}$ where $\phi = (12\ldots n+1)$, $\psi = \prod_{2\leq i<n+3-i}(i \quad n+3-i)$ are generators of D_{n+1}. In the sequel ϕ and ψ will always denote these two permutations. Every D-n-quasigroup can be defined by a system of idetities ([7]). The existence of an n-quasigroup (Q, f) of order mp for every $m > n$, $p \geq 2$ such that $\Pi(f) = D_{n+1}$, follows from [2].

Here we shall continue investigation of D-n-quasigroups and consider several classes of D-n-quasigroups satisfying some additional conditions.

2. Dihedral n-groups

Since every D-n-quasigroup is cyclic, by the results from [8] we get the following theorems.

Theorem 1. *If (Q, f) is an (i,j)-associative D-n-quasigroup, then for every integer m f is (i+m,j+m)-associative n-quasigroup, (where $i+m, j+m$ are reduced modulo n).*

Theorem 2. *Let (Q, f) be an (i,j)-associative D-n-quasigroup, where $j-i$ is relatively prime to n. Then f is an n-group.*

Theorem 3. *Let (Q, f) be an n-group, where $n = 2k$, $k \in N$. The n-group (Q, f) is a D-n-group iff there exists a commutative group $(Q, +)$ such that $x = -x$ for all $x \in Q$ and*

$$f(x_1^n) = \sum_1^n x_i + c$$

where c is a fixed element from Q.

Theorem 4. *There exists a nontrivial finite D-n-group (Q, f) of order q, where n is even, iff $q = 2^t$, $t \in N$. (An n-groupoid (Q, f) is called trivial iff $|Q| = 1$.)*

Theorem 5. *Let (Q, f) be an n-group, where $n = 2k+1$, $k \in N$. The group (Q, f) is a D-n-group iff there exists a commutative group $(Q, +)$ such that*

$$f(x_1^n) = x_1 - x_2 + x_3 - \ldots + x_n + c, \tag{2}$$

where $c = -c$ is an element from Q.

Theorem 6. *A nontrivial finite D-n-group of order q, where $n = 2k + 1$, $k \in N$ exists for every $Q \in N$, and every such n-group is represented by (2).*

When $n = 2k$, $k \in N$, an n-group described in Theorem 3 is an n-group with unity. A unit of that n-group is the element $c \in Q$, and there are no other units.

When $n = 2k + 1$, $k \in N$, then an n-group described in Theorem 5 in the case $c = 0$ is an n-group with unity and every element of that n-group is a unity, and in the case $c \neq 0$ it is an n-group without unity.

In [3] it is proved that the Hosszú-Gluskin theorem is valid for m-associative n-quasigroups, where $m \geq n+2$, which means that every such n-quasigroup is necessarily an n-group. Hence the theorems analogous to Theorems 3 and 5 of the present paper can be proved for dihedral m-associative n-quasigroups, $m \geq n + 2$.

3. Medial D-n-quasigroups

Theorem 7. *Let (Q, f) be a medial n-quasigroup. (Q, f) is dihedral iff there exists a commutative group $(Q, +)$ such that*

a) when $n = 2k$, $k \in N$,

$$f(x_1^n) = -\sum_{i=1}^{n} x_i + b,$$

b) when $n = 2k + 1$, $k \in N$,

$$f(x_1^n) = \theta x_1 - x_2 + \theta x_3 - x_4 + \ldots + \theta x_n + b,$$

where θ is an automorphism of the group $(Q, +)$ such that $\theta^2 = \epsilon$ (identity mapping) and b is a fixed element of Q such that $\theta b = -b$.

Proof. In [1] it is proved that for every medial n-quasigroup (Q, f) there exist a commutative group $(Q, +)$ such that

$$f(x_1^n) = \sum_{i=1}^{n} \theta_i x_i + b,$$

where θ_i, $i = 1, \dots, n$ are automorphisms of the group $(Q, +)$, $\theta_i\theta_j = \theta_j\theta_i$ for all $i, j \in N_n$ and b is a fixed element from Q.

If $f(x_1^n) = x_{n+1}$, then

$$\sum_{i=1}^{n} \theta_i x_i + b = x_{n+1}, \tag{3}$$

and since f is dihedral, $f = f^{\psi}$, hence

$$\theta_1 x_1 + \theta_2 x_{n+1} + \theta_3 x_n + \dots + \theta_n x_3 + b = x_2 \tag{4}$$

Putting in (3) and (4) $x_3 = \dots = x_{n+1} = 0$, where 0 is the neutral element of $(Q, +)$, we get

$$\theta_1 x_1 + \theta_2 x_2 + b = 0 \tag{5}$$

and

$$\theta_1 x_1 + b = x_2. \tag{6}$$

Subtracting (6) from (5) it follows that $\theta_2 x_2 = -x_2$. Since in (3) any n elements can be arbitrarily chosen, it follows that $\theta_2 x = -x$ for all $x \in Q$.

Putting now in (3) and (4) $x_2 = x_4 = \dots = x_{n+1} = 0$, we get

$$\theta_1 x_1 + \theta_3 x_3 + b = 0$$

and

$$\theta_1 x_1 + \theta_n x_3 + b = 0,$$

which implies $\theta_3 = \theta_n$. By a similar procedure we get that $\theta_i = \theta_{n+3-i}$ for all $i = 3, \dots, n$.

Since f is dihedral the following identity is valid

$$f(x_n, f(x_1^n), x_1^{n-2}) = x_{n-1},$$

that is,

$$\begin{aligned} \theta_1 x_n - (\theta_1 x_1 - x_2 + \theta_3 x_3 + \dots + \theta_n x_n + b) \\ + \theta_3 x_1 + \dots + \theta_n x_{n-2} + b = x_{n-1}, \end{aligned} \tag{7}$$

where $\theta_i = \theta_{n+3-i}$ for all $i = 3, \dots, n$.

Putting in (7) $x_2 = \ldots = x_n = 0$, we get $\theta_1 = \theta_3$, then putting $x_1 = x_2 = x_4 = \ldots = x_n = 0$ we get $\theta_3 = \theta_5$ and similarly we obtain that all automorphisms θ_i with odd indices are equal.

Further, putting in (7) $x_1 = x_3 = \ldots = x_n = 0$, it follows that $\theta_4 x = -x$, putting $x_1 = x_2 = x_3 = x_5 = \ldots = x_n = 0$, we get $\theta_6 x = -x$ and similarly we get that for all even indexes i, $\theta_i x = -x$.

From the dihedrality of f it follows that the identity

$$f(f(x_1^n), x_1^{n-1}) = x_n$$

holds, that is,

$$\theta_1(\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n + b) + \theta_2 x_1 + \ldots + \theta_n x_{n-1} + b = x_n.$$

Putting in the preceding identity $x_1 = \ldots = x_n = 0$, we get that $\theta_1 b = -b$, and putting $x_2 = \ldots = x_n = 0$ we get that $\theta_1^2 x = x$ for all $x \in Q$.

When n is even, from $\theta_3 = \theta_n$ it follows that $\theta_i x = -x$ for all $i \in N_n$.

Hence, if we denote $\theta_{2k+1} = \theta$ for all k, we have proved that when n is even

$$f(x_1^n) = -\sum_{i=1}^{n} x_i + b,$$

and when n is odd

$$f(x_1^n) = \theta x_1 - x_2 + \theta x_3 - x_4 + \ldots + \theta x_n + b,$$

where $\theta^2 = \epsilon$ and $\theta b = -b$.

The converse part of the theorem is straightforward.

4. Self-orthogonal D-n-quasigroups

The set $\{(Q, f_1), \ldots, (Q, f_n)\}$ of n-quasigroups is said to be orthogonal iff for each $(a_1^n) \in Q^n$, there exists a unique $(b_1^n) \in Q^n$ such that

$$f_i(b_1^n) = a_i, \quad i = 1, \ldots, n.$$

If (Q, f) is an n-quasigroup such that the set $\{f, f_1, \ldots, f_{n-1}\}$ is otrhogonal, where f_i, $i = 1, \ldots, n-1$ are parastrophes of f defined by $f_i(x_1^n) = f(x_{i+1}^n, x_1^i)$, $i = 1, \ldots, n-1$, then (Q, f) is called a self-orthogonal n-quasigroup. Self-orthogonal cyclic n-quasigroups were considered in [9].

Theorem 8. *If (Q, f) is a commutative group, $n > 2$ odd integer, such that the mapping $x \mapsto 2x$ is a bijection, then by*

$$f(x_1^n) = \sum_{i=1}^{n} (-1)^{i+1} x_i \tag{8}$$

a self-orthogonal D-n-quasigroup (Q, f) is defined.

Proof. It is obvious that $f = f^\phi = f^\psi$, hence (Q, f) is a D-n-quasigroup.

Let a_1^n be arbitrary elements from Q and consider the system

$$\begin{cases} f(x_1^n) = a_1 \\ f_i(x_1^n) = a_{i+1}, \quad i = 1, \ldots, n-1 \end{cases} \tag{9}$$

where the parastrophes f_i, $i = 1, \ldots, n-1$ of the n-quasigroup f are defined by $f_i(x_1^n) = f(x_{i+1}^n, x_1^i)$. Adding the i-th and (i+1)-th equation from (9) for $i = 1, \ldots, n-1$ and the first and the last equation, we get that $x_i = (a_i + a_{i+1})/2$, $i = 1, \ldots, n-1$, $x_n = (a_1 + a_n)/2$ which is a unique solution of (9).

Theorem 9. *If $n, q > 2$ are odd integers, then there exists a self-orthogonal D-n-quasigroup of order q.*

Proof. In the ring Z_q of integers modulo q, the mapping $x \mapsto 2x$ is a bijection, hence using the group $(Z_q, +)$ by (8) we get a self-orthogonal D-n-quasigroup of order q.

If $(Q_1, \oplus_1)$ and $(Q_2, \oplus_2)$ are two commutative groups having the property that $x \mapsto x \oplus_i x$, $i = 1, 2$, are bijections, then their direct product $Q_1 \times Q_2$ has the same property. This means that if by (8) two self-orthogonal D-n-quasigroups of orders q_1 and q_2 respectively, are constructed, then using a direct product a self-orthogonal D-n-quasigroup of order $q_1 q_2$ can be constructed.

5. Menger D-n-quasigroups

If (Q, f) is an n-groupoid, the identity

$$f(x_1^{i-1}, f(y_1^n), x_{i+1}^n) = f(z_1^{i-1}, y_i, z_{i+1}^n), \tag{10}$$

where $z_j = f(x_1^{i-1}, y_j, x_{i+1}^n)$, $j = 1, \ldots, i-1, i+1, \ldots, n$ is called the i-th Menger identity ([1]). Identity (10) for $i = 1$ is called the Menger identity and the corresponding n-groupoid satisfying it is called a Menger n-groupoid.

Theorem 10. *If $n = 2k+1$, $k \in N$, then a D-n-quasigroup (Q, f) satisfying the i-th Menger identity for all $i \in N_n$ is trivial.*

If $n = 2k$, $k \in N$, then there exists a nontrivial D-n-quasigroup (Q, f) of order q satisfying the i-th Menger identity for all $i \in N_n$ iff $q = 2^t$, $t \in N$. This n-quasigroup has the form $f(x_1^n) = x_1 \ldots x_n$, where $(Q, \cdot)$ is a boolean group.

Proof. If (Q, f) is an n-quasigroup satisfying the i-th Menger identity for all $i \in N_n$, then ther exists a commutative group $(G, \cdot)$ of degree $n-2$ such that $f(x_1^n) = x_1 \ldots x_n$ ([1])(a group $(Q, \cdot)$ is of degree m iff $x^m = e$ for all $x \in Q$, where e is the unit of the group).

Let (Q, f) be a D-n-quasigroup satisfying the i-th Menger identity for all $i \in N_n$. If $f(x_1^n) = x_{n+1}$, then

$$x_1 \ldots x_n = x_{n+1},$$

and since $f = f^\phi$, we have

$$x_2 \ldots x_{n+1} = x_1.$$

From the last two equations it follows that

$$(x_1 x_{n+1}^{-1})^2 = e.$$

This means that for every $x \in Q$, $x^2 = e$. Hence if n is odd from the fact that for all $x \in Q$, $x^{n-2} = e$ and $x^2 = e$ it follows that $x = e$ for all $x \in Q$, that is, $|Q| = 1$.

If n is even, since all nonidentity elements from $(Q, \cdot)$ are of order 2, $(Q, \cdot)$ is a boolean group, hence the order of $(Q, \cdot)$ is 2^t, $t \in N$.

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REZIME

NEKE KLASE DIEDARSKIH n-KVAZIGRUPA

n-kvazigrupa (Q, f) se naziva diedarska ako je $f(x_1, \ldots, x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = x_{\sigma(n+1)}$ za svaku permutaciju $\sigma \in D_{n+1}$, gde je D_{n+1} diedarska podgrupa simetrične grupe S_{n+1} stepena $n+1$. Diedarske n-kvazigrupe (D-n-kvazigrupe) predstavljaju generalizaciju totalno simetričnih binarnih kvazigrupa.U ovom radu razmatrane su sledeće klase D-n-kvazigrupa: (i,j)-asocijativne D-n-kvazigrupe, D-n-grupe, medijalne D-n-kvazigrupe, samoortogonalne D-n-kvazigrupe i D-n-kvazigrupe koje zadovoljavaju Mengerove identitete. Odredjene su neke njihove osobine i date neke karakterizacije.

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