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A FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS IN RANDOM NORMED SPACES

Olga Hadžić
Institute of Mathematics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this paper a probabilistic generalization of Göhde's result from [2] is proved.

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1. Introduction

The notion of a random normed space was introduced by A.N.Sherstnev [7] and many fixed point theorems in random normed spaces are obtained ([3], [4], [5]). The aim of this paper is to prove a generalization of the well known result of Göhde about the existence of a fixed point of a nonexpansive mapping in Banach spaces which has a compact iterate.

2. Preliminaries

Let Δ denote the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from the set of real numbers $\mathbf{R}$ into $[0,1]$ so that $\sup_{x\in\mathbf{R}} F(x) = 1$).

A random normed space $(S, \mathcal{F}, t)$ is an ordered triple where S is a real or complex vector space, t is a T-norm which is stronger then the T-norm

$t_m(t \geq t_m$ and $t_m(a,b) = \max\{a+b-1,0\})$ and the mapping $F: S \to \Delta$ satisfies the following conditions, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

a) $F_p = H \Leftrightarrow p = \theta (\theta$ is the neutral element of $S)$,

b) For every $p \in S$, every $u > 0$ and every $r \in K \backslash \{0\}$ (K is the scalar field): $F_{rp}(u) = F_p(\frac{u}{|r|})$.

c) For every $(p,q) \in S \times S$ and every $(u,v) \in (0,\infty) \times (0,\infty)$

$$F_{p-q}(u+v) \geq t(F_p(u), F_q(v)).$$

If the T-norm t is continuous then S is a topological vector space with the fundamental system of neighbourhoods of zero given by

$$U(\epsilon,\lambda) = \{u; u \in S, F_u(\epsilon) > 1-\lambda\}.$$

If $(S,\mathcal{F},t)$ is a random normed space and A a nonempty subset of S the probabilistic diameter of A is the function $D_A(.) : [0,\infty) \to [0,1]$ given by [1]

$$D_A(u) = \sup_{s<u} \inf_{p,q \in A} F_{p,q}(s), \ u \in [0,\infty).$$

A set $A \subseteq S$ is probabilistic bounded if and only if $\sup_{u\in[0,\infty)} D_A(u) = 1$. In [1] and [8] the notions of the Kuratowski and the Hausdorff functions of noncompactness are given.

Definition 1. *Let A be a probabilistic bounded subset of S. The Kuratowski function of noncompactness $\alpha_A(.) : [0,\infty) \to [0,1]$ is given by*
$\alpha_A(u) = \sup\{s \mid s > 0,$ *there is a finite family* $A_j (j \in J)$ *such that* $A = \bigcup_{j\in J} A_j$ *and* $D_{A_j}(u) \geq s$, *for every* $j \in J\}, u \in [0,\infty)$.

The Kuratowski function has the following properties:

1) $\alpha_A \in \Delta$,

2) $\alpha_A(u) \geq D_A(u)$, for every $u \in [0,\infty)$,

3) $\theta \neq A \subset B \subset S \Rightarrow \alpha_A(u) \geq \alpha_B(u)$, for every $u \in [0,\infty)$,

4) $\alpha_{A\cup B}(u) = \min\{\alpha_A(u), \alpha_B(u)\}$, for every $u \in [0,\infty)$,

5) $\alpha_A(u) = \alpha_{\bar{A}}(u)$, for every $u \in [0,\infty)$, where $\bar{A}$ is the closure of A,

6) $\alpha_A = H \Rightarrow A$ is precompact .

Definition 2. *Let A be a probabilistic bounded subset of S. The Hausdorff function of noncompactness $\beta_A(.) : [0,\infty) \to [0,1]$ is given by*

$\beta_A(u) = \sup\{r \mid r > 0$, *there exists a finite subset A_f of S such that* $\tilde{F}_{A,A_f}(u) \geq r\}$ *where for every two probabilistic bounded subsets A and B of S*

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \quad u \in [0,\infty)).$$

The function β has properties 1) - 6) (in which β stands instead of α).

Let $(S, \mathcal{F}, t)$ be a random normed space, K a probabilistic bounded subset of S and $T : K \to S$. If $T(K)$ is probabilistic bounded subset of S and for every $B \subset K$ the following implication holds:

$$\gamma_{T(B)}(u) \leq \gamma_B(u), \text{ for every } \quad u > 0 \Rightarrow B \quad \text{is precompact}$$

where $\gamma_A(.) \in \{\alpha_A(.), \beta_A(.)\}$, we say that the mapping T is γ densifying on the set K.

Definition 3. *Let X be a topological space, $F : X \to X$ and M a nonempty subset of X. The set M is said to be an attractor for F if for every $x \in X$*

$$\overline{\cup_{n\in \mathbf{N}} F^n(x)} \cap M \neq \emptyset.$$

3. A fixed point theorem

D.Göhde proved in [2] that every nonexpansive mapping $T : X \to X$, where X is a Banach space, with the property that $\overline{T^{n_0}(X)}$ is compact for some $n_0 \in \mathbf{N}$, has at least one fixed point.

The next theorem is a probabilistic generalization of Göhde's. **Theorem** *Let $(S, \mathcal{F}, t)$ be a complete random normed space with a continuous T-norm t, A a nonempty, convex and probabilistic bounded subset of S and $T : A \to A$ so that the following conditions are satisfied:*

(i) For every $x, y \in A$ and every $s \in \mathbf{R}^+ = [0, \infty)$

$$F_{Tx-Ty}(s) \geq F_{x-y}(s).$$

(ii) There exists an attractor $M \subset A$ for the mapping T and the mapping T is densifying on M in respect to γ where $\gamma(.) \in \{\alpha(.), \beta(.)\}$. Then there exists $x \in A$ so that $x = Tx$. *Proof.* Let $x_0 \in A$ and $k_n \in (0,1)(n \in N)$ so that $\lim_{n\to\infty} k_n = 1$. For every $n \in N$ and every $x \in A$, let $T_n x = k_n Tx + (1-k_n)x_0$. Then $T_n x \in A$ and

$$F_{T_n x - T_n y}(k_n s) = F_{k_n Tx - k_n Ty}(k_n s) =$$

$$= F_{Tx-Ty}(s) \geq F_{x-y}(s)$$

for every $n \in N$, every $x, y \in A$ and every $s \in \mathbf{R}^+$. From [4] it follows that there exists $x_n \in A$ so that $x_n = T_n x_n$, for every $n \in \mathbf{N}$.

Further

$$F_{x_n - Tx_n}(s) = F_{k_n Tx_n + (1-k_n)x_0 - Tx_n}(s) =$$

$$= F_{(k_n-1)(Tx_n - x_0)}(s) = F_{Tx_n - x_0}\left(\frac{s}{1-k_n}\right).$$

Since A is probabilistic bounded it follows that for every $s \in \mathbf{R}^+$

(1) $$\lim_{n\to\infty} F_{x_n - Tx_n}(s) = 1.$$

Since M is an attractor for T, it follows that for every $n \in \mathbf{N}$ there exists $y_n \in M$ so that

(2) $$y_n \in \overline{\cup_{m\in\mathbf{N}} T^m x_n}.$$

Let us prove that $\lim_{n\to\infty} F_{y_n - Ty_n}(s) = 1$, for every $s \in \mathbf{R}^+$. Let $r \in (0,1)$ and $s > 0$. We shall prove that there exists $n(s,r) \in \mathbf{N}$ so that for every $n > n(s,r) : F_{y_n - Ty_n}(s) > 1 - r$. Since the mapping t is continuous there exists $u \in (0,1)$ so that:

$$1 \geq x, y, z > u \Rightarrow t(x, t(y,z)) > 1 - r.$$

Further, from (2) we conclude that for every $n \in \mathbf{N}$ there exists $m_n \in \mathbf{N}$ such that

$$F_{y_n - T^{m_n} x_n}\left(\frac{s}{3}\right) > u.$$

Since for every $(n,k) \in \mathbf{N} \times \mathbf{N}$:

$$F_{T^k x_n - T^{k+1} x_n}(s) \geq F_{x_n - T x_n}(s)$$

we have that:

$$F_{y_n - T y_n}(s) \geq t(F_{y_n - T^{m_n} x_n}(\frac{s}{3}),$$
$$t(F_{T^{m_n} x_n - T^{m_n+1} x_n}(\frac{s}{3}), F_{T^{m_n+1} x_n - T y_n}(\frac{s}{3})))$$
$$\geq t(F_{y_n - T^{m_n} x_n}(\frac{s}{3}), t(F_{x_n - T x_n}(\frac{s}{3}), F_{T^{m_n} x_n - y_n}(\frac{s}{3}))).$$

Let $n(s,r) \in \mathbf{N}$ be such that for every $n > n(s,r)$

$$F_{x_n - T x_n}(\frac{s}{3}) > u.$$

The existence of such a number $n(s,r)$ follows from (1) .

Then for every $n > n(s,r), F_{y_n - T y_n}(s) > 1 - r$ which implies that

$$\lim_{n \to \infty} F_{y_n - T y_n}(s) = 1.$$

We shall prove that the set $\overline{\{y_n \mid n \in \mathbf{N}\}}$ is compact. Since the mapping T is densifying on M in respect to γ is it enough to prove that for every $u \in \mathbf{R}^+$

(3) $$\gamma_{T[\{y_n | n \in \mathbf{N}\}]}(u) = \gamma_{\{y_n | n \in \mathbf{N}\}}(u).$$

First, we shall suppose that $\gamma = \beta$. In order to prove (3) we shall prove that

(4) $$\beta_{\{y_n | n \in \mathbf{N}\}}(u) \leq \beta_{T[\{y_n | n \in \mathbf{N}\}]}(u), \text{ for every } u \in \mathbf{R}^+$$

and

(5) $$\beta_{T[\{y_n | n \in \mathbf{N}\}]}(u) \leq \beta_{\{y_n | n \in \mathbf{N}\}}(u), \text{ for every } u \in \mathbf{R}^+.$$

Since the function β is left continuous we shall prove that for every $u > 0$ and every $s \in (0,u)$

(6) $$\beta_{\{y_k | k \in \mathbf{N}\}}(u-s) \leq \beta_{\{T y_k | k \in \mathbf{N}\}}(u).$$

Then from (6) we obtain (4). If $\beta_{\{y_k | k \in \mathbf{N}\}}(u-s) = 0$ then (6) holds and hence we shall suppose that $\beta_{\{y_k | k \in \mathbf{N}\}}(u-s) > 0$. In order to prove (6) we shall prove the implication

(7) $$0 < r < \beta_{\{y_k | k \in \mathbf{N}\}}(u-s) \Rightarrow r \leq \beta_{\{T y_k | k \in \mathbf{N}\}}(u).$$

From the inequality $r < \beta_{\{y_k|k\in\mathbf{N}\}}(u-s)$ and the definition of the function β it follows that there exists a finite subset A_f of S such that

$$\bar{F}_{\{y_n|n\in\mathbf{N}\},A_f}(u-s) > r.$$

So, for every $n \in \mathbf{N}$ there exists $z(n) \in A_f$ such that $F_{y_n,z(n)}(u-s) > r$. We shall prove that for every $\delta_1 \in (0,r)$

$$\beta_{\{Ty_k|k\in\mathbf{N}\}}(u) \geq r - \delta_1. \tag{8}$$

Since $t(1,r) = r$ from the continuity of the mapping t it follows that there exists $\delta_2 \in (0,1)$ so that

$$1 \geq h > 1 - \delta_2 \Rightarrow t(h,r) > r - \delta_1.$$

From the relation $\lim_{n\to\infty} F_{y_n - Ty_n}(s) = 1$, for every $s > 0$ it follows that there exists $n(s,\delta_2) \in \mathbf{N}$ such that $F_{y_k - Ty_k}(\frac{s}{2}) > 1-\delta_2$ for every $k > n(s,\delta_2)$ and so

$$F_{Ty_k - z(k)}(u - \frac{s}{2}) \geq t(F_{Ty_k - y_k}(\frac{s}{2}), F_{y_k - z(k)}(u-s)) \geq$$

$$\geq t(F_{Ty_k - y_k}(\frac{s}{2}), r) > r - \delta_1.$$

Since $\beta_{\{Ty_k|k\in\mathbf{N}\}}(u) = \beta_{\{Ty_k|k>n(s,\delta_2)\}}(u)$ we obtain (8) and so the inequality $r \leq \beta_{\{Ty_k|k\in\mathbf{N}\}}(u)$ holds. Similarly we can prove that

$$\beta_{\{y_k|k\in\mathbf{N}\}}(u) \geq \beta_{\{Ty_k|k\in\mathbf{N}\}}(u)$$

and so

$$\beta_{\{y_k|k\in\mathbf{N}\}}(u) = \beta_{\{Ty_k|k\in\mathbf{N}\}}(u)$$

for every $u \in \mathbf{R}$. Since T is densifyng on M in respect to β we conclude that the set $\overline{\{y_k \mid k \in \mathbf{N}\}}$ is compact. Suppose that $\gamma = \alpha$ and prove that for every $u \in \mathbf{R}$ and $s \in (0,u)$

$$\alpha_{\{y_k|k\in\mathbf{N}\}}(u-s) \leq \alpha_{\{Ty_k|k\in\mathbf{N}\}}(u). \tag{9}$$

Let $0 < r < \alpha_{\{y_k|k\in\mathbf{N}\}}(u-s)$. We shall prove that for every $\delta_1 \in (0,r)$

$$r - \delta_1 \leq \alpha_{\{Ty_k|k\in\mathbf{N}\}}(u) \tag{10}$$

which implies (9) .

Using the definition of the function α we obtain that there exists a finite family $\{Y_1, Y_2, ..., Y_n\}(Y_i \subset S,\ i \in \{1,2,...,n\})$ such that $\{y_k \mid k \in \mathbf{N}\} = \cup_{i=1}^{n} Y_i$ where

$$(11) \qquad D_{Y_i}(u-s) > r, \text{ for every } \ i \in \{1,2,...,n\},$$

This implies that for every $x, y \in Y_i, F_{x-y}(u-s) > r$. Let $\delta \in (0,r)$. Further, let $\delta_2 \in (0,1)$ be such a number that the following implication holds:

$$1 \geq v, w > 1-\delta_2 \Rightarrow t(v, t(r,w)) > r - \delta_1.$$

Since $t(1, t(r,1)) = r$ such an element δ_2 exists.

Let, for every $j \in \{1,2,...,n\}, Z_j$ be a subset of S which is defined by

$$Z_j = \{z \mid z \in S,\ F_{z-y}(\frac{s}{4}) > 1-\delta_1 \text{ for some } \ y \in Y_j\}.$$

If $n(s,\delta_2) \in \mathbf{N}$ is such that for every $k > n(s,\delta_2)$

$$F_{y_k - Ty_k}(\frac{s}{4}) > 1-\delta_2$$

then

$$\{Ty_k \mid k > n(s,\delta_2)\} \subset \cup_{j=1}^{n} Z_j.$$

It can be proved that $D_{Z_j}(u) > r-\delta_1$ for every $j \in \{1,2,...,n\}$ and so

$$\alpha_{\{Ty_k \mid k > n(s,\delta_2)\}}(u) = \alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u) \geq r - \delta_1.$$

This proves (9) and so $\alpha_{\{y_k \mid k \in \mathbf{N}\}}(u) \leq \alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u)$, for every $u \in \mathbf{R}$. Similarly we can prove that

$$\alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u) \leq \alpha_{\{y_k \mid k \in \mathbf{N}\}}(u)$$

and so $\alpha_{\{y_k \mid k \in \mathbf{N}\}}(u) \leq \alpha_{\{Ty_k \mid k \in \mathbf{N}\}}(u)$, for every $u \in \mathbf{R}$.

Since the mapping T is γ densifying on M it follows that the set $\overline{\{y_n \mid n \in \mathbf{N}\}}$ is compact. If $\lim_{k\to\infty} y_{n_k} = y$ then from $\lim_{n\to\infty} y_n - Ty_n = 0$ it follows that $Ty = y$ since the mapping T is continuous.

Corollary *Let $(S, \mathcal{F}, t)$ be a complete random normed space with a continuous T-norm t, A a nonempty, closed, convex and probabilistic bounded subset of S and T a probabilistic nonexpansive mapping of A into A (i.e.(i) from Theorem holds). If there exists $n_0 \in \mathbf{N}$ such that $\overline{T^{n_0}(A)}$ is compact* *then there exists $x \in A$ such that $x = Tx$.*

Proof . We shall prove that all the conditions of the Theorem are satisfied, where $M = \overline{T^{n_0-1}(A)}$. It is obvious that M is an attractor for T. Suppose that B is an arbitrary subset from the set $\overline{T^{n_0-1}(A)}$ such that

$$\gamma_{T(B)}(u) \leq \gamma_B(u), \text{for every} \quad u > 0.$$

From the continuity of the mapping T it follows that $T(B) \subset \overline{T(T^{n_0-1}(A))}$ and so

$$1 = \gamma_{\overline{T^{n_0}(A)}}(u) \leq \gamma_{T(B)}(u) \leq \gamma_B(u), \text{ for every } u > 0.$$

This means that $\gamma_B = H$ which implies that B is precompact and hence T is $\gamma-$ densifying on M.

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REZIME

TEOREMA O NEPOKRETNOJ TAČKI ZA NEEKSPANZIVNA PRESLIKAVANJA U SLUČAJNIM NORMIRANIM PROSTORIMA

U radu je dokazana verovatnosna generalizacija Göhdeovog rezultata iz [2].

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