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A COMMON FIXED POINT THEOREM FOR A FAMILY OF MAPPINGS IN CONVEX METRIC SPACES

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Abstract

In this paper a generalization of the common fixed point theorem from [2] is proved.

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1. Introduction

There are many fixed point theorems or common fixed point theorems in convex metric spaces ([1], [2], [3], [4], [7], [8]).

In [2] we proved the following common fixed point theorem.

Theorem 1. *Let (M, d) be a complete, convex metric space, K a nonempty, closed subset of M, $f, S, T : K \to M$ continuous mappings so that $\partial K \subseteq SK \cap TK$, $f(K) \cap K \subseteq SK \cap TK$ and*

$$Tx \in \partial K \Rightarrow f(x) \in K; \quad Sx \in \partial K \Rightarrow fx \in K.$$

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If (f,S) and (f,T) are weakly commutative and there exists a nondecreasing function $q:[0,\infty)\to[0,1)$ such that

$$d(fx,fy)\le q(d(Sx,Ty))d(Sx,Ty), \text{for every } x,y\in K$$

then there exists $z\in K$ so that

$$z=fz\in\{Tz,Sz\}.$$

If $S,T:M\to M$ then there exists one and only one $z\in K$ such that $z=fz=Tz=Sz$.

In this paper we shall prove a generalization of Theorem 1 if $S,T:M\to M$.

The notion of a weakly commutative pair of mappings is introduced by Sessa in [6] and the notion of a compatible pair of mappings by Jungck in [5]. There are examples of compatible pairs which are not weakly commutative and weakly commutative pairs which are not commutative.

The next definition is a slight modification of the Jungck definition.

Definition 1. *Let (M,d) be a metric space, $A:\mathcal{D}(A)\to M, S:\mathcal{D}(S)\to M$, $\mathcal{D}(A)\subseteq M, \mathcal{D}(S)\subseteq M$. The pair (A,S) is said to be compatible if for every sequence $\{x_n\}_{n\in\mathbf{N}}$ from $\mathcal{D}(A)\cap\mathcal{D}(S)$ such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=t\in M$ and $Ax_n\in\mathcal{D}(S), Sx_n\in\mathcal{D}(A)$, for every $n\in\mathbf{N}$ the relation*

$$\lim_{n\to\infty}d(ASx_n,SAx_n)=0$$

holds.

2. A common fixed point theorem

The following fixed point theorem is a generalization of Theorem 1 if $S,T:M\to M$.

Theorem 2. *Let (M,d) be a complete, convex metric space, K a nonempty closed subset of $M, S,T:M\to M$ continuous mappings so that $\partial K\subseteq SK\cap TK$, for every $i\in\mathbf{N}$, $A_i:K\to M$ continuous mappings such that*

$A_iK \cap K \subseteq SK \cap TK$, (A_i, S) and (A_i, T) compatible pairs and there exists a nondecreasing function $q : [0, \infty) \to [0, 1)$ such that

$$d(A_ix, A_jy) \leq q(d(Sx, Ty))d(Sx, Ty),$$

for every $i \neq j (i, j \in \mathbf{N})$ and every $x, y \in K$.

If for every $i \in \mathbf{N}$ and $x \in K$ the implications

$$Tx \in \partial K \Rightarrow A_ix \in K; \quad Sx \in \partial K \Rightarrow A_ix \in K$$

hold, then there exists $z \in K$ such that

$$z = Tz = Sz = A_iz, \text{ for every } \ i \in \mathbf{N}.$$

Proof . Let $x \in \partial K$ and $p_0 \in K$ so that $x = Tp_0$. Then for every $i \in \mathbf{N}$, $A_ip_0 \in K$ and so $A_1p_0 \in SK$ which implies that there exists $p_1 \in K$ such that $Sp_1 = A_1p_0 \in K$. Let $p'_1 = A_1p_0, p'_2 = A_2p_1$. If $p'_2 \in K$ then there exists $p_2 \in K$ so that $Tp_2 = A_2p_1$ and if $p'_2 \notin K$, since M is a convex metric space, there exists $p_2 \in K$ such that $Tp_2 \in \partial K$ and

$$d(Sp_1, Tp_2) + d(Tp_2, A_2p_1) = d(Sp_1, A_2p_1).$$

If we continue this process we obtain sequences $\{p_n\}_{n \in \mathbf{N}}$ and $\{p'_n\}_{n \in \mathbf{N}}$ such that for every $n \in \mathbf{N}$, $p_n \in K$, $p'_{n+1} = A_{n+1}p_n$ and the following implications hold:

(i) $p'_{2n} \in K \Rightarrow p'_{2n} = Tp_{2n}$,
$p'_{2n} \notin K \Rightarrow Tp_{2n} \in \partial K$ and

$$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, A_{2n}p_{2n-1}) =$$

$$= d(Sp_{2n-1}, A_{2n}p_{2n-1})$$

(ii) $p'_{2n+1} \in K \Rightarrow p'_{2n+1} = Sp_{2n+1}$,
$p'_{2n+1} \notin K \Rightarrow Sp_{2n+1} \in \partial K$ and

$$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, A_{2n+1}p_{2n}) = d(Tp_{2n}, A_{2n+1}p_{2n}).$$

Let us prove that there exists $z \in K$ such that $z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}$. The sets P_0, P_1, Q_0, Q_1 will be defined in the following way:

$$P_0 = \{p_{2n}; p'_{2n} = Tp_{2n}, \ n \in \mathbf{N}\},$$

$$P_1 = \{p_{2n}; p'_{2n} \neq Tp_{2n},\ n \in \mathbf{N}\},$$
$$Q_0 = \{p_{2n+1}; p'_{2n+1} = Sp_{2n+1},\ n \in \mathbf{N}\},$$
$$Q_1 = \{p_{2n+1}; p'_{2n+1} \neq Sp_{2n+1},\ n \in \mathbf{N}\}.$$

It is easy to see that we have the following possibilities:

$$(p_{2n}, p_{2n+1}) \in P_0 \times Q_0; (p_{2n}, p_{2n+1}) \in P_0 \times Q_1;$$
$$(p_{2n}, p_{2n+1}) \in P_1 \times Q_0.$$

a) $(p_{2n}, p_{2n+1}) \in P_0 \times Q_0$.
Then

$$d(Tp_{2n}, Sp_{2n+1}) = d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq q[d(Sp_{2n-1}, Tp_{2n})]d(Sp_{2n-1}, Tp_{2n}).$$

b) $(p_{2n}, p_{2n+1}) \in P_0 \times Q_1$.
Then

$$d(Tp_{2n}, Sp_{2n+1}) = d(Tp_{2n}, A_{2n+1}p_{2n}) - d(Sp_{2n+1}, A_{2n+1}p_{2n})$$
$$\leq d(Tp_{2n}, A_{2n+1}p_{2n}) = d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq q[d(Sp_{2n-1}, Tp_{2n})]d(Sp_{2n-1}, Tp_{2n}).$$

c) $(p_{2n}, p_{2n+1}) \in P_1 \times Q_0$.
Then

$$d(Tp_{2n}, Sp_{2n+1}) \leq d(Tp_{2n}, A_{2n}p_{2n-1}) + d(A_{2n}p_{2n-1}, Sp_{2n+1})$$
$$= d(Tp_{2n}, A_{2n}p_{2n-1}) + d(A_{2n}p_{2n-1}, A_{2n+1}p_{2n}) \leq$$
$$\leq d(Tp_{2n}, A_{2n}p_{2n-1}) + q[d(Sp_{2n-1}, Tp_{2n})]d(Sp_{2n-1}, Tp_{2n})$$
$$\leq d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, A_{2n}p_{2n-1}) =$$
$$= d(Sp_{2n-1}, A_{2n}p_{2n-1}).$$

Since $p_{2n} \in P_1$ implies that $p_{2n-1} \in Q_0$ we have that $Sp_{2n-1} = A_{2n-1}p_{2n-2}$ and so

$$d(Tp_{2n}, Sp_{2n+1}) \leq d(Sp_{2n-1}, A_{2n}p_{2n-1}) =$$
$$= d(A_{2n-1}p_{2n-2}, A_{2n}p_{2n-1}) \leq q[d(Tp_{2n-2}, Sp_{2n-1})].$$
$$.d(Tp_{2n-2}, Sp_{2n-1}).$$

It can be proved similarly that the following implications hold:

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_0 \Rightarrow d(Sp_{2n-1}, Tp_{2n}) \leq$$

$$\leq q[d(Tp_{2n-2}, Sp_{2n-1})].d(Tp_{2n-2}, Sp_{2n-1});$$

$$(p_{2n-1}, p_{2n}) \in Q_1 \times P_0 \Rightarrow d(Sp_{2n-1}, Tp_{2n}) \leq$$

$$q[d(Tp_{2n-2}, Sp_{2n-3})]d(Tp_{2n-2}, Sp_{2n-3});$$

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_1 \Rightarrow d(Sp_{2n-1}, Tp_{2n}) \leq$$

$$\leq q[d(Tp_{2n-2}, Sp_{2n-1})]d(Tp_{2n-2}, Sp_{2n-1}).$$

It is easy to prove that

$$d(Tp_{2n}, Sp_{2n+1}) \leq [q(\delta)]^{n-1}.\delta, \tag{1}$$

$$d(Sp_{2n+1}, Tp_{2n+2}) \leq [q(\delta)]^{n}\delta, \tag{2}$$

where $\delta = \max\{d(Tp_2, Sp_3), d(Tp_2, Sp_1)\}$.

Since $q(\delta) < 1$, (1) and (2) imply that sequences $\{Tp_{2n}\}_{n\in\mathbf{N}}$ and $\{Sp_{2n+1}\}_{n\in\mathbf{N}}$ are Cauchy sequences in K. Since M is complete there exists $z \in K$ so that $z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}$. Further, there exists at least one subsequence $\{Tp_{2n_k}\}_{k\in\mathbf{N}}$ or $\{Sp_{2m_k+1}\}_{k\in\mathbf{N}}$ such that for every $k \in \mathbf{N}$, $p_{2n_k} \in P_0$ or $p_{2m_k+1} \in Q_0$. Hence, let us suppose that $p_{2n_k} \in P_0$, $k \in \mathbf{N}$. From the definition of the set P_0 it follows that $Tp_{2n_k} = A_{2n_k}p_{2n_k-1}$, $k \in \mathbf{N}$.

We shall prove that for every $m \in \mathbf{N}$

$$d(z, A_m z) \leq q(L)d(z, Tz) \tag{3}$$

for a positive number L. Since $\lim_{k\to\infty} d(Sp_{2n_k-1}, Tz) = d(z, Tz)$, there exists $L > 0$ such that

$$d(Sp_{2n_k-1}, Tz) \leq L, \quad k \in \mathbf{N}.$$

Since $Tp_{2n_k} = A_{2n_k}p_{2n_k-1}$, $k \in \mathbf{N}$ we have for $2n_k \neq m$:

$$d(Tp_{2n_k}, A_m z) = d(A_{2n_k}p_{2n_k-1}, A_m z) \leq q[d(Sp_{2n_k-1}, Tz)]\times$$

$$\times d(Sp_{2n_k-1}, Tz) \leq q(L)d(Sp_{2n_k-1}, Tz).$$

When $k \to \infty$ we obtain that (3) holds. We shall prove that $\lim_{k\to\infty} A_m p_{2n_k} = z$. Suppose that $2n_k \neq m$.
Then

$$d(A_m p_{2n_k}, Tp_{2n_k}) = d(A_m p_{2n_k}, A_{2n_k} p_{2n_k-1}) \leq$$
$$\leq q(L')d(Tp_{2n_k}, Sp_{2n_k-1})$$

where $d(Tp_{2n_k}, Sp_{2n_k-1}) \leq L'$, $k \in \mathbf{N}$. Hence $\lim_{k\to\infty} d(A_m p_{2n_k}, Tp_{2n_k}) = 0$ which implies that $\lim_{k\to\infty} A_m p_{2n_k} = z$. Since $Tp_{2n_k} \in K$ and $T : M \to M$ from the relation $\lim_{k\to\infty} d(A_m p_{2n_k}, Tp_{2n_k}) = 0$ it follows, using the compatibility of A_m and T, that

$$\lim_{k\to\infty} d(T(A_m p_{2n_k}), A_m(Tp_{2n_k})) = 0.$$

This implies that $Tz = A_m z$, for every $m \in \mathbf{N}$ since T and A_m are continuous. From (3) we obtain that $d(z, A_m z) \leq q(L)d(z, A_m z)$, for every $m \in \mathbf{N}$ and since $q(L) < 1$ we have that $z = A_m z$. It remains to be proved that $z = Sz$. Suppose that $s \neq m$. Then we have for every $k \in \mathbf{N}$ that

$$d(A_s p_{2n_k-1}, A_m p_{2n_k}) \leq q(L')d(Sp_{2n_k-1}, Tp_{2n_k})$$

and since $\lim_{k\to\infty} d(Sp_{2n_k-1}, Tp_{2n_k}) = 0$ we obtain that $\lim_{k\to\infty} A_s p_{2n_k-1} = z$, for every $s \in \mathbf{N}$. We shall prove that $\lim_{k\to\infty} A_m Sp_{2n_k-1} = Sz$. Since (A_m, S) are compatible and $\lim_{k\to\infty} Sp_{2n_k-1} = \lim_{k\to\infty} A_m p_{2n_k-1} = z$ it follows that

$$\lim_{k\to\infty} d(A_m Sp_{2n_k-1}, SA_m p_{2n_k-1}) = 0.$$

Since S is continuous and $\lim_{k\to\infty} A_m p_{2n_k-1} = z$ we obtain that $\lim_{k\to\infty} A_m Sp_{2n_k-1} = Sz$. Further, we have that for $m \neq s$ and every $k \in \mathbf{N}$

$$d(A_s p_{2n_k}, A_m Sp_{2n_k-1}) \leq q(L")d(Tp_{2n_k}, SSp_{2n_k-1}))$$

which implies, when $k \to \infty$ that $d(z, Sz) \leq q(L")d(z, Sz)$. Here $L"$ is such that $d(Tp_{2n_k}, S(Sp_{2n_k-1})) \leq L"$ for every $k \in \mathbf{N}$. Since $q(L") < 1$ it follows that $z = Sz$ and so

$$z = Tz = Sz = A_m z, \text{ for every } m \in \mathbf{N}.$$

If z is in K then it is enough to suppose that $T, S : K \to M$ since from the relations $\lim_{k\to\infty} A_m p_{2n_k-1} = \lim_{k\to\infty} A_m p_{2n_k} = z$ it follows that there exists $k_0 \in \mathbf{N}$ such that

$$A_m p_{2n_k-1} \in K, \quad A_m p_{2n_k} \in K, \quad k \geq k_0$$

and every $m \in \mathbf{N}$.

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REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI ZA FAMILIJU PRESLIKAVANJA U KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazano je uopštenje teoreme o zajedničkoj nepokretnoj tački iz rada [2].

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