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CONFORMAL DIFFEOMORPHISM BETWEEN TWO f – MANIFOLDS

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Abstract

f-manifolds have been studied by various authors including Blair D. [1], Endo H. [2], Yano K. [5] and the integrability conditions of this structure [6]. Singh K. and Srivastava R. in [4] prove a number of theorems involving the fundamental tensor for almost Hermitian manifolds with torsion and study a torsion preserving conformal diffeomorphism.

The paper studies the conformal diffeomorphism between two Riemannian f-manifolds and the Nijenhuis tensor on such manifolds. The result is obtained: The structure f' on $\mathcal{M}'^n$ is integrable iff and only iff the structure f on $\mathcal{M}^n$ is integrable, where $\mathcal{M}^n$ and $\mathcal{M}'^n$ are conformal diffeomorphically Riemannian f-manifolds with the structure tensors f and f', respectively.

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1. Introduction

Let $\mathcal{M}^n$ be a C^∞ real differentiable manifold, $\mathcal{R}(\mathcal{M}^n)$ the ring of real valued differentiable functions over $\mathcal{M}^n$, $\mathcal{H}(\mathcal{M}^n)$ the module of derivatives of

$\mathcal{R}(\mathcal{M}^n)$. Then $\mathcal{H}(\mathcal{M}^n)$ is a Lie algebra over the real numbers and elements of $\mathcal{H}(\mathcal{M}^n)$ are called vector fields.

A C^∞ $n-$dimensional differentiable manifold is called an $f-$manifold iff there exists a non-null tensor field f of type $(1,1)$, of constant rank r, $r \leq n$, such that $f^3 + f = 0$. The $(1,1)$ tensor fields ℓ and m defined by

$$\ell = -f^2, \quad m = I + f^2$$

are complementary projection operators, where I denotes the identity operator. It is easily seen that $\ell^2 = \ell$, $m^2 = m$, $\ell m = m\ell = 0$, $\ell + m = I$.

Let L and M be complementary distributions corresponding to ℓ and m respectively. $Dim\ L = r$, $dim\ M = n - r$. Then:

$$\begin{array}{ll} \ell f = f\ell = f, & fm = mf = 0, \\ f^2\ell = \ell f^2 = -\ell, & f^2 m = mf^2 = 0. \end{array}$$

It can be seen from the above relations that f acts as an almost complex structure on L and as a null operator on M.

An $f-$manifold $\mathcal{M}^n$ always admits a positive definite Riemannian metric g such that

$$g(X,Y) = g(fX,fY) + g(mX,Y). \tag{1.1}$$

The above metric satisfies the following relations:

$$g(mX,Y) = g(X,mY), \quad g(X,mY) = g(mX,mY),$$

$$g(fX,Y) = g(f^2X,fY), \quad g(X,fY) = g(fX,f^2Y).$$

Let ∇ be the Riemannian connexion with respect to the metric g on $\mathcal{M}^n$. Thus,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1.2}$$

$$[X,Y] = \nabla_X Y - \nabla_Y X.$$

Let $\mathcal{N}$ denote the Nijenhuis tensor of f i.e.

$$\begin{aligned} [f,f](X,Y) &= \mathcal{N}(X,Y) = \\ = [fX,fY] - f[fX,Y] &- f[X,fY] + f^2[X,Y]. \end{aligned} \tag{1.3}$$

2. Conformal diffeomorphism between two f-manifolds

Let $\mathcal{M}^n$ and $\mathcal{M}'^n$ be two manifolds with structure tensors f and $f'(f^3+f=0,\ f'^3+f'=0)$, and admitting the Riemannian metric g and g' respectively as in (1.1).

Let $\phi:\mathcal{M}^n \to \mathcal{M}'^n$ be a diffeomorphism. For $X \in \mathcal{H}(\mathcal{M}^n)$, let $X' \equiv \phi_* X$ where ϕ_* is the Jacobian map or the differential of ϕ [3]. Then, ϕ is called a conformal diffeomorphism, if there exists some real valued function $\sigma \in \mathcal{R}(\mathcal{M}^n)$ such that

$$g'(X',Y')\circ\phi = e^{2\sigma}g(X,Y)\ ;\ X,Y \in \mathcal{H}(\mathcal{M}^n).$$

For a real valued function φ, $grad\varphi$ is defined by [4] in the following way

$$g(grad\varphi, X) = X(\varphi);\quad X \in \mathcal{H}(\mathcal{M}^n).$$

We shall be using the following Lemma [4].

Lemma 1. *Let $\phi:\mathcal{M}^n \to \mathcal{M}'^n$ be a conformal diffeomorphism on two C^∞ manifolds $\mathcal{M}^n$ and $\mathcal{M}'^n$ have ∇ and ∇' as Riemannian connexions. Then*

$$\nabla'_{X'}Y' = \{\nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)grad\sigma\}'. \tag{2.1}$$

Let us assume that ϕ preserves the f-structure i.e. $f'X' = (fX)'$.

Theorem 1.

$$\begin{aligned}(\nabla'_{X'}f')(Y') &= \{(\nabla_X f)(Y) + (fY)(\sigma)Y - Y(\sigma)(fX) +\\ &\quad + g(fX,Y)grad\sigma + g(X,Y)f\ grad\sigma\}'.\end{aligned}$$

Proof. From (2.1), we have

$$(\nabla'_{X'}f')(Y') = \nabla'_{X'}(f'Y') - f'(\nabla'_{X'}Y') =$$

$$= \{\nabla_X(fY) + X(\sigma)fY + fY(\sigma)X - g(X,fY)grad\sigma\}' -$$

$$-f'((\nabla_X Y) + X(\sigma)Y + Y(\sigma)X - g(X,Y)grad\sigma)' =$$

$$= ((\nabla_X f)(Y) + fY(\sigma)X - Y(\sigma)fX + g(fX,Y)grad\sigma + g(X,Y)fgrad\sigma)'$$

which proves the theorem. □

Theorem 2. *Let $\mathcal{N}$ and $\mathcal{N}'$ be Nijenhuis tensors, of f on $\mathcal{M}^n$ and of f' on $\mathcal{M}'^n$ respectively. Then we have*

$$(\mathcal{N}(X,Y))' = \mathcal{N}'(X',Y').$$

Proof. From (1.3) and the previous theorem we have:

$$\begin{aligned}
\mathcal{N}'(X',Y') &= [f'X',f'Y'] - f'[X',f'Y'] - f'[f'X',Y'] + f'^2[X',Y'] = \\
&= \nabla'_{(f'X')}(f'Y') - \nabla'_{(f'Y')}(f'X') - f'\nabla'_{X'}(f'Y') + f'\nabla'_{(f'Y')}X' - \\
&\quad -f'\nabla'_{(f'X')}Y' + f'\nabla'_{Y'}(f'X') + f'^2\nabla'_{X'}Y' - f'^2\nabla'_{Y'}X' = \\
&= (\nabla'_{f'X'}f')(Y') - (\nabla'_{f'Y'}f')(X') - f'(\nabla'_{X'}f')(Y') + f'(\nabla'_{Y'}f')(X') = \\
&= ((\nabla_{fX}f)(Y) + fY(\sigma)fX - Y(\sigma)f^2X + g(f^2X,Y)grad\sigma + \\
&\quad +g(fX,Y)fgrad\sigma - (\nabla_{fY}f)(X) - fX(\sigma)fY + X(\sigma)f^2Y - \\
&\quad -g(f^2Y,X)grad\sigma - g(fY,X)fgrad\sigma - f(\nabla_X f)(Y) - fY(\sigma)fX + \\
&\quad +Y(\sigma)f^2X - g(fX,Y)fgrad\sigma - g(X,Y)f^2grad\sigma + f(\nabla_Y f)(X) + \\
&\quad +fX(\sigma)fY - X(\sigma)f^2Y + g(fY,X)fgrad\sigma + g(Y,X)f^2grad\sigma)' = \\
&= (\mathcal{N}(X,Y))',
\end{aligned}$$

which proves the theorem. □

The following result [6] is very well known. The structure f is integrable iff and only iff $[f,f]=0$.

The consequence of Theorem 2 is the following theorem:

Theorem 3. *Let $\phi : \mathcal{M}^n \to \mathcal{M}'^n$ be a structure preserving a conformal diffeomorphism between two $f-$manifolds $\mathcal{M}^n$ and $\mathcal{M}'^n$. The structure f' on $\mathcal{M}'^n$ is integrable iff and only iff the structure f on $\mathcal{M}^n$ is integrable.*

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REZIME

KONFORMNO DIFEOMORFNO PRESLIKAVANJE IZMEDJU DVE f–MNOGOSTRUKOSTI

U radu je ispitivano konformno difeomorfno preslikavanje izmedju dve f-–mnogostrukosti i veza izmedju Nijenhuisovih tenzora takvih mnogostrukosti.

Struktura f' na mnogostrukosti $\mathcal{M}'^n$ je integrabilna ako i samo ako je struktura f na $\mathcal{M}^n$ integrabilna, gde su $\mathcal{M}^n$, $\mathcal{M}'^n$ konformno difeomorfne Rimanove f–mnogostrukosti sa tenzorima strukture f i f' respektivno.

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