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WEAK CONTINUITY AND ALMOST CONTINUITY FOR MULTIFUNCTIONS

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Abstract

The purpose of the note is to investigate some properties of weakly continuous multifunctions and almost continuous multifunctions and to obatin new characterizations of almost continuous multifunctions using the notion of a semi-open set generalizing the results from [7] - [9].

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Weak continuous multifunctions are defined by us in [11] as a generalization of univocal weak continuous applications defined by Levine in [6]. Smithson also defines weak continuous multifunctions in [22]. The properties and some applications of weak continuous multifunctions are given in [1],[2],[11],[12],[14],[16] and [18].

Almost continuous multifunctions are defined by us in [13] as a generalization of univocal almost continuous applications defined by Singal in [20]. The properties of almost continuous multifunctions are studied in [4],[15],[16] and [17].

Let X and Y be two topological spaces. For a multifunction $F : X \longrightarrow Y$, we shall denote by $F^+(G)$ and $F^-(G)$ the upper and lower inverse of the set $G \subset Y$ and thus

$$F^+(G) = \{x \in X : F(x) \subset G.\}; F^-(g) = \{x \in X : F(x) \cap G \neq 0\}.$$

As the paracompact sets can be defined in two way [3], we shall define the paracompact sets in the following way:

Definition 1. *The set M in the topological space X is called strictly paracompact if every covering of M by open sets from X can be refined by a covering by open sets from X, locally finite in X [3].*

It is known from [3] that the notion of the strictly paracompact set and that of the paracompact set defined with the help of the relative topology do not coincide.

Definition 2. *Let X and Y be two topological spaces.*

(a) The multifunction $F : X \longrightarrow Y$ is upper weakly continuous (u.w.c.) respectively, upper almost continuous (u.a.c.) in the point $x_0 \in X$ if for every open set $G \subset Y$ with $F(x_0) \subset G$, there exists an open set $V \subset X$ containing x_0, se that $F(V) \subset ClG$, respectively, $F(V) \subset IntClV$ ($Cl\, G$ stands for the closure of G).

(b) The multifunction $F : X \longrightarrow Y$ is lower weakly continuous (l.w.c.) respectively, lower almost continuous (l.a.c.) in the point $x_0 \in X$ if for every open set $G \subset Y$ with $F(x_0) \cap G \neq 0$, there exists an open set $V \subset X$ conataining x_0, so that $F(x) \cap G \neq 0$, $\forall x \in V$, respectively, $F(x) \cap IntClG \neq 0, \forall x \in V$.

(c) The multifunction $F : X \longrightarrow Y$ is weakly continuous (w.c.), respectively, almost continuous (a.c.), in the point $x_0 \in X$ if F is upper and lower weakly continuous, respectively, upper and lower almost continuous, in the point $x_0 \in X$.

(d) The multifunctional $F : X \longrightarrow Y$ is weakly continuous (u.w.c.,l.w.c.), respectively, almost continuous (u.a.c.,l.a.c.) if it has this property in any point $x \in X$ [11],[13],[22].

The following implications hold

1. u.s.c. $\Rightarrow$ u.a.c. $\Rightarrow$ u.w.c.

2. l.s.c. $\Rightarrow$ u.a.c. $\Rightarrow$ l.w.c. .

If the range Y is a regular space and $F(x), \forall x \in X$ is a strictly paracompact set then

(1') u.s.c. $\Leftrightarrow$ u.a.c. $\Leftrightarrow$ u.w.c.

If the range Y is a regular space, then

(2') l.s.c. $\Leftrightarrow$ l.a.c. $\Leftrightarrow$ l.w.c. [11], [13], [14], [22]

Definition 3. *A topological space X is almost regular if for any point $x \in X$ and each regular closed set $A \subset X$ with $x \overline{\in} A$ there exist disjoint open sets U and V such that $x \in U$ and $A \subset V$ [21].*

Every regular space is almost regular, but the converse is not always true [21].

Definition 4. *A topological is said to be almost compact if every open cover admits a finite subfamily, the closures of whose members cover the space [19].*

Theorem 1. *Let X and Y be two topological spaces. If $F : X \longrightarrow Y$ is a surjective multifunction with an X compact space and Y T_4 space, so that:*

1) F is punctually compact.
2) F is u.w.c.,

then F is u.a.c.

Proof. Let $V = \{V_i : i \in I\}$ be any open cover of $Y.F(x)$ being compact $\forall x \in X$, there exists a finite subfamily $V' = \{V_{i_k} : k = 1, ..., n\}$ of V such that $F(x) \subset \bigcup_{k=1}^{n} V_{i_k}$. Let $V_x = \bigcup_{k=1}^{n} V_{i_k}$, then $F(x) \subset V_x$. The family $\{V_x : x \in X\}$ is then an open cover of $Y.F$ being v.w.c.. By Theorem 6, implication $(1) \Rightarrow (2)$ from [11] it follows that $F^+(V_x)c$ Int $F^+(ClV_x)$. F being a surjectiove multifunction the family $\{F^+(V_x)\}$ is a covering of Y and the family $\{Int\ F^+(ClV_x) : x \in X\}$ is an open cover of Y. Since X is compact it has a finite family $\{IntF^+(ClV_{x_j}) : j = 1, 2, .., m\}$ such that $X = \bigcup_{j=1}^{m} IntF^+(ClV_{x_j})$. F being a surjection, then

$$Y = F(X) = F(\bigcup_{j=1}^{m} IntF^+(ClV_{x_j}) \subset \bigcup_{j=1}^{m} F(F^+(ClV_{x_j}) \subset \bigcup_{j=1}^{m} ClV_{x_j}.$$

Hence we have $Y = \bigcup_{j=1}^{m} ClV_{x_j}$. This implices that Y is almost compact. Every almost compact and T_4-space is almost regular [10]. pp.139).Thus, by Theorem 2.3 of [16], F is u.a.c.

Corollary 1. *If f is a weakly-continuous univocal mapping of a compact space X onto a T_4-space, then f is almost continuous ([8], Theorem 3).*

Theorem 2. *If the multifunction $F_1 : X \longrightarrow Y$ is u.s.c. and the multifunction $F_2 : Y \longrightarrow Z$ is u.a.c., then the multifunction $F = F_2 \circ F_1 : X \longrightarrow Z$ is u.a.c.*

Proof. Let G be any regularly open set of Y. Then $F^+(G) = F_1^+(F_2^+(G))$. According to Theorem 2.4, implication (1) $\Rightarrow$ (3) of [13], $F_2^+(G)$ is an open set of Y. Then, $F^+(G)$ is an open set of Y because F_1 is u.s.c. and F is u.a.c. from Theorem 2.4, implication (3) $\Rightarrow$ (1) of [13].

Theorem 3. *If the multifunction $F_1 : X \longrightarrow Y$ is l.s.c. and the multifunction $F_2 : Y \longrightarrow Z$ is l.a.c., then multifunction $F = F_2 \circ F_1 : X \longrightarrow Z$ is — a.c.*

The proof is similar to the proof of Theorem 2 and follows from Theorem 2.2 of [13].

Let X be a topological space. The family of regularly open sets of X forms a base for a semi-regular topology which is called the semi-regularization of the topology of X [19. In the present note we shall denote by X_S the set X endowed with such a topology.

Theorem 4. *If $F : X \longrightarrow Y$ is a multifunction so that:*

1) Y is an almost regular space,

2) F is punctually strictly paracompact,

then F is weakly continuous if and only if $F : X_S \longrightarrow Y$ is almost continuous.

Proof. Necessity. Suppose F is weakly continuous. Let V be any regularly open set of Y and $x \in F^+_{X_s}(V) = F^+(V)$, namely $F(x) \subset V$.

Let $F(x) = \bigcup_{i \in I}\{y_i\}$. Y being an almost regular space, for the regular open set V and for every y_i there is a regular open set V_i, so that $y_i \in V_i \subset ClV_i \subset V$ according to Theorem 2.2(a) of [21]. So, we have

$$F(x) \subset \bigcup_{i \in I} V_i \subset \bigcup_{i \in I} ClV_i \subset V.$$

$F(x)$ being a strictly paracompact set, there is a family $A = \{A_j : j \in J\}$ of open sets, so that $A_j \subset V_i$ for some $i \in I$ and A is a local finite covering of $F(x)$. We shall have

$$F(x) \subset \bigcup_{j \in J} A_j \subset \bigcup_{i \in I} V_i \subset \bigcup_{i \in I} ClV_i \subset V$$

and

$$F(x) \subset \bigcup_{j \in J} A_j \subset \bigcup_{j \in J} ClA_j \subset \bigcup_{i \in I} ClV_i \subset V.$$

Let $V_0 = \bigcup_{j\in J} A_j$. The family A being locally finite, then $ClV_0 = \bigcup_{j\in J} ClA_j$, so $F(x) \subset V_0 \subset ClV_0 \subset V$.

Since every u.w.c. multifunction, punctually strictly paracompact into an almost regular space, is u.a.c. from Theorem 2.3 of [16], thus by Theorem 2.4, implication (1) $\Rightarrow$ (2) of [13] it follows that

$$x \in F^+(V_0) \subset IntF^+(IntClV_0) \subset F^+(ClV_0) \subset F^+(V).$$

Int $F^+(IntClV_0)$ is an open set and $F^+(ClV_0)$ is a closed set from Theorem 2.2, implication (1) $\Rightarrow$ (6) of [13] because F being l.w.c. from Theorem 2.4 of [16] F is l.a.c. and ClV_0 being a regularly closed set.

We shall have

$$x \in F^+(V_0) \subset IntF^+(IntClV_0) \subset$$

$$\subset Int\ ClF^+(IntClV_0) \subset F^+(ClV_0) \subset F^+(V).$$

Let $U = IntClF^+(IntClV_0)$ be. Then, U is a regularly open set of X and

$$x \in F^+(V_0) \subset U \subset F^+(ClV_0) \subset F^+(V) = F^+_{X_S}(V).$$

This implies that $F^+(V)$ is an open set of X_S. Therefore, by Theorem 2.4, implication (3) $\Rightarrow$ (1) of [13], F_{X_S} is u.a.c.

Then we can prove that F_{X_s} is l.a.c. Let V be any regularly open set of Y and $x \in F^-_{X_S}(V) = F^-(V)$, then $F(x \cap V \neq 0$. From Theorem 2.2 of [21], there is an open set $G \subset Y$ such that $F(x) \cap G \neq \phi$ and $ClG \subset V$. Thus,

$$x \in F^-(G) \subset F^-(ClG) \subset F^-(V).$$

The multifunction F being l.w.c., from Theorem 2.4 of [16], it follows that F is l.a.c.. Then, by Theorem 2.2 of [13], it follows that

$$x \in F^-(G) \subset IntF^-(IntClG) \subset F^-(ClG) \subset F^-(V).$$

Int $F^+(IntClG)$ is an open set and $F^+(ClG)$ is a closed set from Theorem 2.4, implication (1) $\Rightarrow$ (6) of [13] because F being u.w.c. and punetually strictly paracompact from Theorem 2.3 of [16] is u.a.c. and $Cl\,G$ being a regularly closed set. We shall have

$$x \in F^-(G) \subset IntF^-(IntClG) \subset IntClF^-(IntClG) \subset F^-(ClG) \subset F^-(V)$$

Let $U = IntClF^-(IntClG)$ be.Thus U is a regularly open set and

$$x \in F^-(G) \subset U \subset F^-(ClG) \subset F^-(V) = F^-_{X_S}(V).$$

This implies that $F^-(V)$ is an open set of X_S. Therefore, by Theorem 2.2, impication (3) $\Rightarrow$ (1) of [13], F_{X_S} is l.a.c.

Sufficiency. Suppose F_{X_S} is almost continuous. Since the identity mapping $i_X : X \longrightarrow X_S$ is a continuous function (multifunction) and $F = F_{X_S} \circ i_X$, then, by Theorem 2 and 3, F is almost continuous.Therefore, F is weakly continuous from (1) and (2).

Corollary 2. *Let Y be an almost regular space. Then, a univocal mapping $f : X \longrightarrow Y$ is weakly continuous if and only if $f_{X_S} : X_S \longrightarrow X$ is almost continuous ([9], Theorem 1)*

Theorem 5. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent:*

1. *F is u.w.c.*
2. *For each open set $G \subset Y, ClF^-(G) \subset F^-(ClG)$ [18].*

Theorem 6. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent:*

1. *F is l.w.c.*
2. *For each open set $G \subset Y, ClF^+(G) \subset F^+(ClG)$ [18].*

Now, we shall prove two similar theorems for almost continuous multifunctions.

Definition 5. *Let X be a topological space. A set $V \subset X$ is said to be semi-open if there exists an open set $G \subset X$ such that $G \subset V \subset ClG$ [6].*

Theorem 7. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent:*

1. *F is u.a.c.*
2. *For each semi-open set $G \subset Y, ClF^-(G) \subset F^-(ClG)$.*

Proof. (1) $\Rightarrow$ (2). Let G be a semi-open set of Y, then $Cl\,G$ is a closed set and according to theorem 2.4, implication (1) $\Rightarrow$ (5) of [13], we have

$$F^-(ClG) \supset ClF^-(ClIntClG) \supset ClF^-(ClIntG).$$

The set G being semi-open, then from Theorem 1 of [6], $G \subset ClIntG$, therefore,

$$ClF^-(ClIntG) \supset ClF^-(G)$$

and

$$ClF^{-}(G) \subset F^{-}(ClG).$$

(2) $\Rightarrow$ (1). Since every regularly closed set is semi-open, therefore, for every regularly closed set G, we have

$$ClF^{-}(G) \subset F^{-}(ClG) = F^{-}(G).$$

Therefore, $F^{-}(G)$ is closed and F is u.a.c. from Theorem 2.4, implication (6) $\Rightarrow$ (1) of [13].

Theorem 8. *For a multifunction $F : X \longrightarrow Y$ the following are equivalent*

1. *F is l.a.c.*
2. *For each semi-open set $G \subset Y, ClF^{+}(G) \subset F^{+}(ClG)$.*

The proof is similar to the proof of Theorem 7 and follows from Theorem 2.2 of [13].

Corollary 3. *A univocal application $f : X \longrightarrow Y$ is almost continuous if and only if $Clf^{-1}(G) \subset f^{-1}(ClG)$ for each semi-open set $G \subset Y$. ([7], Theorem 2.1).*

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REZIME

SLABA NEPREKIDNOST I GOTOVO NEPREKIDNOST ZA MULTIFUNKCIJE

Svrha ove note je da se istraže neke osobine slabo neprekidnih multifunkcija i i skoro neprekidnih multifunkcija i da se dobiju nove karakterizacije skoro neprekidnih multifunkcija korišćenjem pojma poluotvorenog skupa, čime se uopštavaju rezultati iz [7]-[9].

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