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A FAMILY OF EXPONENTIAL SPLINE DIFFERENCE SCHEMES

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Abstract

Using a spline in tension for the problem: $-\varepsilon y'' + p(x)y = f(x)$, $0 < x < 1$, $y(0) = \alpha_0$, $y(1) = \alpha_1$, $0 < \varepsilon << 1$, a family of difference schemes is derived. The schemes have a second order of uniform convergence. Some of them converge with respect to ε.

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1. Introduction

We shall consider the problem

$$(1)\qquad \begin{cases} -\varepsilon y'' + p(x)y = f(x),\ 0 < x < 1, \\ y(0) = \alpha_0,\ \ y(1) = \alpha_1, \end{cases}$$

where $0 < x < 1$, $p(x)$ and $f(x)$ are smooth functions and $p(x) \geq p > 0$.

It is known that problem (1) has a unique solution y, which in general displays boundary layers at $x = 0$ and $x = 1$. The follownig lemma describes some properties of the exact solution $y = y(x)$.

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Lemma 1 ([1]). *Let* $y(x) \in C^4[0,1]$ *and* $p'(0) = p'1) = 0$. *The the solution of* (1) *can be written in the form*

$$y(x) = u(x) + w(x) + g(x),$$

where

$$\begin{aligned} u(x) &= q_0 \exp(-x(p(0)/\varepsilon)^{1/2}) \\ w(x) &= q_1 \exp(-(1-x)(p(1)/\varepsilon)^{1/2}) \end{aligned}$$

q_0, q_1 *are bounded functions of* ε *independent of* x *and*

$$| q^{(i)}(x) | \leq M(1 + \varepsilon^{1-(1/2)i}), \;\; i = 0(1)4,$$

M *is a constant independent of* ε. □

According to Lemma 1 we have

$$y(x) = \begin{cases} u(x) + g_1(x), & 0 < x \leq 1/2, \\ w(x) + g_2(x), & 1/2 < x \leq 1, \end{cases}$$

where

$$| g_j^{(i)}(x) | \leq M(1 + \varepsilon^{1-(1/2)i}), \;\; i = 0(1)4, \;\; j = 1,2.$$

Taking into account this in [3] the collocation method via the spline in tension for problem (1) id derived. A boundary layer functions $u(x)$ and $w(x)$ are introduced into the base of the spline. The uniform convergence of the second order is achived. In [4] the corresponding difference scheme is analysed. The optimal order of the convergence in the sense of [1] is proved. In this paper we formed differnt difference schemes via a spline in tension which satisfies diffe

rent collocation conditions. They are consequence of a different approxiamation of functions $p(x)$ and $f(x)$ (piecewise constant and piecewise linear). Some of them heve the optimal error estimate in the sense of [1], i.e.

$$| y_i - v_i | \leq Mh \min(h, \sqrt{\varepsilon}), \tag{2}$$

where $y_i = y(x_i)$, v_i is the approximate value for y_i, $x_i = ih$, $i = 0(1)n+1$, $h = 1/n+1$, M is a constant independent of ε and h.

The corresponding results for the cubic spline are given in [6].

Throughout the paper M denotes the positive constants that may take different values in different formulas, but are always independent of ε and h. All the constants in the asymptotic equalities are independent of ε and h.

2. Derivation of the schemes

We seek the solution of problem (1) in the form of the spline in tension $e(x)$, on the mesh $x_i = ih$, $i = 0(1)n+1$, $h = 1/n+1$. On each interval $[x_j, x_{j+1}]$, $e(x)$ has the form

$$e(x) = e_j(x) = v_{j+1}t + v_j(1-t) + \frac{d_{j+1}}{\rho_j^2}\left(\frac{sh\mu_j t}{sh\mu_j} - t\right) + \tag{3}$$

$$\frac{d_j}{\rho_j^2}\left(\frac{sh\mu_j(1-t)}{sh\mu_j} - (1-t)\right),$$

where $\mu_j = \rho_j h$, $t = \frac{x-x_j}{h}$, $x \in [x_j, x_{j+1}]$.

The function $e(x)$ belong to $C^2[0,1]$ and $e_j(x) \in$ span $\{1, x, \exp(-\rho_j x), \exp(\rho_j x)\}$.
The values ρ_j are tension paramenters wich will be detemined. The unknown coefficients d_j and d_{j+1} will be determined so that the function $e(x)$ satisfies the "comparison" problem:

$$\begin{cases} -\varepsilon e''(x) + \tilde{p}(x)e(x) = \tilde{f}(x), \\ e(0) = \alpha_0, \;\; e(1) = \alpha_1, \end{cases} \tag{4}$$

at the grid points. Here $\tilde{p}(x)$ a $\tilde{f}(x)$ are piecewise polinomial approximations to $p(x)$ and $f(x)$.

Let index j be fixed. Starting with piecewise constant approxiamtions we denote the approximate value for $p(x)$ by p^+ when $x \in [x_j, x_{j+1}]$ and by p^- when $x \in [x_j, x_{j-1}]$. Analogously we denote the approximation for $f(x)$. According to (3) and (4) we put

$$
\begin{aligned}
d_{j+1}: &= d_j^+ = -\frac{f^+ - p^+ v_{j+1}}{\varepsilon} \quad \text{for} \quad x \in [x_j, x_{j+1}] \text{ and} \\
d_: &= d_j^- = -\frac{f^- - p^- v_{j+1}}{\varepsilon} \quad \text{for} \quad x \in [x_{j-1}, x_j].
\end{aligned}
$$

Using $\rho_j = \rho^- = \sqrt{p^-/\varepsilon}$ for $x \in [x_{j-1}, x_j]$ and $\rho_j = \rho^+ = \sqrt{p^+/\varepsilon}$ for $x \in [x_j, x_{j+1}]$, from equation

$$
e'_{j-1}(x_j) = e'_j(x_j) \tag{5}
$$

and the boundary conditions, we obtain the scheme

$$
\tilde{R}v_j = \tilde{Q}f_j, \quad j = 1(1)n, \tag{6}
$$

where

$$
\begin{aligned}
\tilde{R}v_j &= R^- v_{j-1} + R^c v_j + R^+ v_{j+1}, \\
\tilde{Q}f_j &= Q^- f^- + Q^+ f^+, \\
R^- &= \frac{1}{h} + \frac{p^-}{\varepsilon(\rho^-)^2}\left(-\frac{\rho^-}{sh\mu^-} + \frac{1}{h}\right) \\
R^+ &= \frac{1}{h} + \frac{p^+}{\varepsilon(\rho^+)^2}\left(-\frac{\rho^+}{sh\mu^+} + \frac{1}{h}\right)
\end{aligned}
$$

$$
R^c = \frac{p^-}{\varepsilon(\rho^-)^2}\left(\rho^- cth\mu^- - \frac{1}{h}\right) + \frac{p^+}{\varepsilon(\rho^+)^2}\left(\rho^+ cth\mu^+ - \frac{1}{h}\right) + \frac{2}{h}
$$

$$
\begin{aligned}
Q^- &= \frac{1}{\varepsilon(\rho^-)^2}\left(\rho^- cth\mu^- - \frac{\rho^-}{sh\mu^-}\right), \\
Q^+ &= \frac{1}{\varepsilon(\rho^+)^2}\left(\rho^+ cth\mu^+ - \frac{\rho^+}{sh\mu^+}\right), \\
v_0 &= \alpha_0, \quad v_{n+1} = \alpha_1, \quad \mu^- = \rho^- h, \quad \mu^+ = \rho^+ h.
\end{aligned}
$$

The chose of approximation to $p(x)$ and $f(x)$ determines the particular scheme.

Let $p^+ = p^- = p(x_j)$ and $f^- = f^+ = f(x_j) = f_j$. Then scheme (6) obtains the form

(7) $$r^- v_{j-1} + r^c v_j + r^+ v_{j-1} = q^c f_j,$$

where

$$r^- = r^+ = -\frac{\rho_j}{sh\ \mu_j},\ \ r^c = 2\ \rho_j\ cth\ \mu_j,\ \ \rho_j = \sqrt{p_j/\varepsilon},\ \ \mu_j = \rho_j h,$$

$$q^c = \frac{1}{\sqrt{\varepsilon p_j}}(2\ cth\ \mu_j - \frac{2}{sh\ \mu_j}).$$

If we consider ρ_j as a tension parameter independent of collocation conditions and if $\rho_j \to 0$, the scheme (7) gets the form

(8) $$-\varepsilon\ h^{-2}(v_{j-1} - 2v_j + v_{j-1}) = f(x_j).$$

Then the spline in tension becomes a cubic one and scheme (8) is derived in [6] via the cubic spline.

Let $f^- = f_{j-1}$, $f^+ = f_{j+1}$, $p^- = p_{j-1}$, and $p^+ = p_{j+1}$. Now scheme (6) has the form

(9) $$r^- v_{j-1} + r^c v_j + r^+ v_{j-1} = q^- f_{j-1} + q^+ f_{j+1},$$

$$\begin{aligned}
r^- &= -\frac{\sqrt{p_{j-1}/\varepsilon}}{sh\ \mu_{j-1}},\ \ r^+ = -\frac{\sqrt{p_{j+1}/\varepsilon}}{sh\ \mu_{j+1}},\\
r^c &= \sqrt{p_{j-1}/\varepsilon}\ cth\ \mu_{j-1} + \sqrt{p_{j+1}/\varepsilon}\ cth\ \mu_{j+1},\\
q^- &= \frac{1}{p_{j-1}}(\sqrt{p_{j-1}/\varepsilon}\ cth\ \mu_{j-1} - \frac{\sqrt{p_{j-1}/\varepsilon}}{sh\ \mu_{j-1}}),\\
q^+ &= \frac{1}{p_{j+1}}(\sqrt{p_{j+1}/\varepsilon}\ cth\ \mu_{j+1} - \frac{\sqrt{p_{j+1}/\varepsilon}}{sh\ \mu_{j+1}}).
\end{aligned}$$

In the case $f^+ = (f_{j+1} + f_j)/2$, $f^- = (f_{j-1} + f_j)/2$, $p^+ = (p_{j+1} + p_j)/2$, $p^- = (p_{j-1} + p_j)/2$, the correspondong scheme is the derived in [2]:

$$(10)\qquad r^- v_{j-1} + r^c v_j + r^+ v_{j-1} = q^- f_{j-1} + q^c f_j + q^+ f_{j+1},$$

$$r^- = -\frac{\rho_{j-1}}{sh\,\mu_{j-1}},\quad r^+ = -\frac{\rho_j}{sh\,\mu_j},\quad r^c = \rho_{j-1} cth\,\mu_{j-1} + \rho_j cth\,\mu_j,$$

$$\begin{aligned} \rho_j &= \sqrt{(p_{j+1}+p_j)/2\varepsilon},\quad \mu_j = h\,\rho_j,\\ q^- &= \frac{1}{2\varepsilon\,\rho_{j-1}}\,(cth\,\mu_{j-1} - \frac{1}{sh\,\mu_{j-1}}),\\ q^+ &= \frac{1}{2\varepsilon\,\rho_{j+1}}\,(cth\,\mu_{j+1} - \frac{1}{sh\,\mu_{j+1}}),\quad q^c = q^- + q^+. \end{aligned}$$

In [2] it is proved that scheme (10) has a second order of the uniform convergence.

When $\tilde{p}(x) = \frac{p_j - p_{j-1}}{h}\,(x - x_{j-1}) + p_{j-1}$ for $x \in [x_{j-1}, x_j]$ equation (5) leads to the scheme derived in [5]:

$$(11)\qquad r^- v_{j-1} + r^c v_j + r^+ v_{j-1} = q^- f_{j-1} + q^c f_j + q^+ f_{j+1},$$

$$\begin{aligned} r^- &= -\frac{\mu_{j-1}}{sh\,\mu_{j-1}},\ r^+ = -\frac{\mu_{j+1}}{sh\,\mu_{j-1}},\ r^c = 2\mu_j\,cth\,\mu_j,\\ q^- &= -\frac{1}{p_{j-1}}\,(1 - \frac{\mu_{j-1}}{sh\,\mu_{j-1}}),\ q^+ = -\frac{1}{p_{j+1}}\,(1 - \frac{\mu_{j+1}}{sh\,\mu_{j+1}}),\\ q^c &= -\frac{2}{p_j}\,(-1 + \mu_j\,cth\mu_j),\ \mu_j = h\,\sqrt{p_j/\varepsilon}. \end{aligned}$$

For scheme (11) estimate (2) is valid ([5]).

When ε is fixed and $h \to 0$, the mentioned schemes become within $0(h^4)$ the schemes derived in [6] via the spline without the fitting factors.

3. Convergence of the schemes

The following theorem gives sufficient conditions for the convergence of schemes (6).

Theorem 1 *Let in (1)* $y(x) \in c^4[0,1]$ *and* $p'(0) = p'(1) = 0$. *Let* v_j *be the approxiamtion to* $y(x_j)$ *obtained using scheme (6) where*

$$\begin{aligned} p^- &= p(x_j - \alpha h) + 0(h^2), \;\; p^+ = p(x_j + \alpha h) + 0(h^2), \\ f^- &= f(x_j - \alpha h) + 0(h^2), \;\; f^+ = f(x_j + \alpha h) + 0(h^2), 0 \leq \alpha \leq 1. \end{aligned}$$

Then, the estimate

(12) $$| \, y(x_j) - v_j \, | \; \leq \; Mh^2$$

holds.

Proof. Let A be a matrix corresponding to scheme (6). Then

(13) $$| \, y(x_j) - v_j \, | \; \leq \; \| \, A^{-1} \, \| \, \max_j \, | \, \tilde{\tau}_j(y) \, |$$

where

$$\begin{aligned} \tilde{\tau}_j(y) &= \tilde{R}y_j - \tilde{R}v_j = \tilde{R}y_j - \tilde{Q}f_j = \tilde{R}y_j - Qf_j + Qf_j - \tilde{Q}f_j, \\ Qf_j &= q^+(-\varepsilon y"_j + p^+ y_j) + q^-(-\varepsilon y"_j + p^- y_j). \end{aligned}$$

After some Taylor developments, we obtain that

(14) $$| \, Qf_j - \tilde{Q}f_j \, | \; \leq \; \begin{cases} Mh^3/\varepsilon & \text{for } \; h^2 \leq \varepsilon \\ Mh^2/\sqrt{\varepsilon} & \text{for } \; \varepsilon \leq h^2. \end{cases}$$

According to Lemma 1, we have

(13) $$\tau_j(y) = \tilde{R}y_j - Qf_j = \tau_j(u) + \tau_j(w) + \tau_j(g),$$

Now, we shall estimate separately the contribution to the error from functions u, w and g. Let $\varepsilon \leq h^2$. Then

$$\tau_j(g) = T_0 g_j + T_1 g_{j'} + T_2 g_{j''} + T_3 g_{j'''} + R^- g^{iv}(\sigma_1) + R^+ g^{iv}(\sigma_2),$$

where $$x_{j-1} < \sigma_1 < x_j < \sigma_2 < x_{j+1},$$

$$\begin{aligned} T_0 &= R^- + R^c + R^+ - Q^- p^- - Q^+ p^+ = 0, \;\; T_1 = -hR^- + hR^+, \\ T_2 &= h^2/2(R^- + R^+) + \varepsilon(Q^- + Q^+), \;\; T_3 = h^3/3! \, (-R^- + R^+). \end{aligned}$$

By Taylor developments and Lemma 1 we obtain

$$(14) \qquad |\, \tau_j(g) \,| \leq Mh^3/\varepsilon.$$

One can check that $\tau_j(u) = 0$ for $p(x) = p(0) = \text{const}$. Thus, if denote

$$\tau_j(u) = \tau_j(u, p^-, p^+),$$

we have $\tau_j(u, p(0), p(0)) = 0$ and $\tau_j(u) = \tau_j(u, p^-, p^+) - \tau_j(u, p(0), p(0))$. Using Taylor expansions about $p(0)$ we obtain

$$(15) \qquad |\, \tau_j(u) \,| \leq Mh^3/\varepsilon.$$

Further, $\tau_j(w, p(1), p(1)) = 0$ where $\tau_j(w) = \tau_j(w, p^+, p^-)$. In a similar way as for $\tau_j(u)$, we can prove than

$$(16) \qquad |\, \tau_j(w) \,| \leq Mh^3/\varepsilon.$$

Since

$$(17) \qquad \|\, A^{-1} \,\| \leq \begin{cases} Mh^3/\varepsilon & \text{for } h^2 \leq \varepsilon \\ Mh^2/\sqrt{\varepsilon} & \text{for } \varepsilon \leq h^2, \end{cases}$$

from (11) - (17) we obtain the statement of Theorem 1 when $h^2 \leq \varepsilon$.

Let $h^2 \geq \varepsilon$. Since $|R^-|, |R^+| \leq M/h$, $|q^-|, |q^+| \leq M/\sqrt{\varepsilon}$, from the expression

$$\tau_j(g) = T_0 g_j + T_1 g_{j'} + R^- h^2 g_{j''}(\xi_1) + R^+ h^2 g_{j''}(\xi_2) + \varepsilon(Q^- + Q^+) g_j'',$$

we obtain

$$(18) \qquad |\, \tau_j(g) \,| \leq Mh.$$

Further,

$$\tau_j(u) = \tau_j(u, p, p) - \tau_j(u, p(0), p(0)) = (R^- - R^-(p(0)))u_{j-1} +$$

$$+ (R^c - R^c(p(0), p(0)))u_j + (R^+ - R^+(p(0)))u_{j+1}$$

where $R^- = R^-(p^-)$, $R^+ = R^+(p^+)$, $R^c = R^c(p^-, p^+)$. Using the known properties of exponential functions [1], we have

$$(19) \qquad |\, \tau_j(u) \,| \leq M\varepsilon/h.$$

Similarly $|\, \tau_j(w) \,| \leq M\varepsilon/h$ and from (13), (18), (19) and (20) we obtain the statement of Theorem 1.

Theorem 2 *Let the assumptions of Theorem 1 be fulfilled. Let v_j be the approximate value for $y(x_j)$ obtained using scheme (7). Then*

$$| y(x_j) - v_j | \leq M \min(h^2, \varepsilon), \quad j = 0(1)n+1.$$

Proof. In the case $h^2 \leq \varepsilon$ the statement follows from Theorem 1 ($\alpha = 0$ and the constant in the asymptotic member is equal to zero). For $\varepsilon \leq h^2$ we have that $Qf_j - \tilde{Q}f_j = 0$. Since $|r^-|, |r^+| \leq M\sqrt{\varepsilon}$ and $| \tau_j(g) | \leq M\sqrt{\varepsilon}$, following the proof of the Theorem 1 we obtain that Theorem 2 holds.

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REZIME

FAMILIJA EKSPONENCIJALNIH SPLAJN DIFERENCNIH ŠEMA

Koristeći splajn u tenziji za problem $-\varepsilon y'' + p(x)y = f(x)$, $0 < x < 1$, $y(0) = \alpha_0$, $y(1) = \alpha_1$, $0 < \varepsilon << 1$, izvedena je familija diferencnih šema. Šeme imaju drugi red uniformne konvergencije. Neke od njih konvergiraju u odnosu na ε.

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