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ON S - QUASI - CONTINUOUS MULTIVALUED MAPS

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Abstract

Among various generalizations of continuity we can see two types; in one of them it is assumed that the inverse images of open sets belong to some class larger than the topology, this way leads - in particular - to quasi - continuous maps [11, 17]. In the second one the inverse images of sets from some subfamily of the given topology are considered. Then, as an example there are s-continuous maps [13, 14, 15,]. Joinging both the above methods the class of s-quasi-continuous maps is introduced.

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1. Basic definitions and properties

For a subset A of a topological space X the symbols $Cl(A)$ and $Int(A)$ are used to denote the closure and the interior of A.
A set $A \subset X$ is said to be:
- semi-open if $A \subset Cl\, Int(A)$, [17],
- semi-closed if $X \backslash A$ is semi-open [2, 3].
The sets $Scl(A) = \bigcap\{B : A \subset B,$ B is semi-closed$\}$

and $Sint(A) = \bigcup\{B : B \subset A,$ B is semi-open$\}$ are called the semi-closure and the semi-interior of A respectively [2, 3].

Now, let F be a multivalued map defined on a topological space X and having values in the non-empty subsets of a topological space Y ; for the simplicity we will write $F : X \to Y$. For any set $V \subset Y$ we denote $F^+(V) = \{x \in X : F(x) \subset V\}$ and $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ [1].

A multivalued map $F : X \to Y$ is called upper (lower) s - quasi - continuous at a point $x_0 \in X$, if for each open set $V \subset Y$ having the connected complement and $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$), there exists a semi- open set $U \subset X$ such that $x_0 \in U \subset F^+(V)$ ($x_0 \in U \subset F^-(V)$). A map F is called upper (lower) s - quasi - continuous if it has this property

at each point.

If, in the above definitions, we omit the condition that $Y \setminus V$ is connected, then we shall obtain the definitions of the upper and the lower quasi - continuity respectively [19, 20].

Any function $f : X \to Y$ can be considered as the multivalued map F given by $F(x) = \{f(x)\}$. In this case we have s - quasi - continuous (quasi - continuous [11, 17]) maps.

The standard proofs of the following two theorems we shall omit.

1.1 Theorem.

A multivalued map $F : X \to Y$ is upper (lower) s - quasi - continuous at $x_0 \in X$ if and only if for each open set $V \subset Y$ with the connected complement and $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$), and for each neighbourhood U of x_0 there exists an open non-empty set $U_1 \subset U$ such that $U_1 \subset F^+(V)$ (resp. $U_1 \subset F^-(V)$).

1.2 Theorem.

For a multivalued map $F : X \to Y$ the following statements are equivalent:

(a) F is upper (lower) s-quasi - continuous;

(b) for every open set $V \subset Y$ having the connected complement the set

$F^+(V)$ (resp. $F^-(V)$) is semi - open in X;
(c) for every connected closed set $M \subset Y$ the set $F^-(M)$ (resp. $F^+(M)$) is semi - closed;
(d) for every set $B \subset Y$ having the connected closure, the following inclusion holds: $Scl(F^-(B)) \subset F^-(Cl(B))$ (resp. $Scl(F^+(B)) \subset F^+(Cl(B))$);
(e) for every set $A \subset Y$ such that $Y \setminus Int(A)$ is connected, $F^+(Int(A)) \subset Sint(F^+(A))$ (resp. $F^-(Int(A)) \subset Sint(F^-(A))$).

Now, let (Y,T) be a topological space and let ς denote the collection of all the open sets whose complements are connected. Let T^* denote the topology on Y generated by taking ς as a subbase. Obviously, $T^* \subset T$.
For any map with values in Y, it holds that:
quasi - continuity $\rightarrow$ T^*-quasi - continuity $\rightarrow$ s - quasi - continuity
and none of the above implications can be reversed neither for multivalued maps nor for functions.

1.3 Examples.

(a) Let R be the set of real numbers with the natural topology and $x_0 \in R$ an established point. The multivalued map $F : R \rightarrow R$ given by

$$F(x) = \begin{cases} [1,2] & \text{for } x < x_0, \\ [3,4] & \text{for } x = x_0, \\ [4,5] & \text{for } x > x_0, \end{cases}$$

is upper and lower s - quasi - continuous but it is neither upper nor lower T^* - quasi - continuous at x_0.
(b) Let X be the set of real numbers with the natural topology and (Y,T) the Sorgenfrey line [4, p. 39]. Then T^* is the co-finite topology on Y [22, Prop. 1]. Let a, b, x_0 be established numbers with $x_0 < a < b$. We define the multivalued map $F : X \rightarrow Y$ assuming

$$F(x) = \begin{cases} [a,b] & \text{for } x = x_0, \\ \{x\} & \text{for } x \neq x_0. \end{cases}$$

The map F is upper and lower T^* - quasi - continuous but it is not upper nor lower quasi - continuous.

2. Points of continuity

For a multivalued map F, the symbols $C^+(F), C^-(F)$ are used to denote the sets of all the points at which F is upper or lower semi - continuous respectively.

A multivalued map $F : X \to Y$ is called upper (lower) almost continuous at a point $x_0 \in X$ if for each open set $V \subset Y$ with $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) it holds that $x_0 \in IntCl(F^+(V))$, (resp. $x_0 \in IntCl(F^-(V))$), [21]. For single valued maps each of these definitions imply almost continuity in the sense of Husain [9].

2.1 Theorem.

Let X be a topological space, Y a locally connected regular one and let $F : X \to Y$ be a multivalued map.
(a) If F is lower s - quasi - continuous, $F(x)$ is a connected compact for $x \in X$ and F is upper almost continuous at x_0, then F is upper semicontinuous at x_0.
(b) If F is upper s - quasi - continuous, lower almost continuous at x_0, then F is lower semicontinuous at x_0.

Proof. Assume that F is not upper semicontinuous at x_0. Then, there exists an open set $V \subset Y$, $F(x_0) \subset V$ such that each neighbourhood U' of x_0 contains a point x' for which $F(x') \not\subset V$. Because Y is regular locally connected and $F(x_0)$ is a connected compact set, we can choose an open connected set V_1 satisfying

$$F(x_0) \subset V_1 \subset Cl(V_1) \subset V.$$

The set $U = IntCl(F^+(V_1))$ is a neighbourhood of x_0, so it contains a point x_1 such that $F(x_1) \cap (Y \setminus Cl(V_1)) \neq \emptyset$.
Using the lower s - quasi - continuity of F at x_1, we choose an open non-empty set $U_1 \subset U$ such that $F(x) \cap (Y \setminus Cl(V_1)) \neq \emptyset$ for $x \in U_1$. But $U_1 \cap F^+(V_1) \neq \emptyset$, thus for some $x_2 \in U_1$ we have $F(x_2) \subset V_1$, which is impossible. Hence, the proof of (a) is completed.
The proof of (b) is analogus.

2.2 Lemma.

Let X be a topological space, Y a second countable one, and let $F : X \to Y$ be a multivalued map. Then:
(a) The set of all the points at wich F is not lower almost continuous is of the first category [12].
(b) If $F(x)$ is compact for each $x \in X$, then the set of all the points at wich F is not upper almost continuous is of the first category [5, Th. 2.1.]

Combining this lemma with Theorem 2.1 we immediately obtain

2.3 Theorem.

Let X be a topological space, Y a locally connected second countable regular one and let $F : X \to Y$ be a multivalued map. Then:
(a) If F is lower s - quasi - continuous and $F(x)$ is connected compact for each $x \in X$, then $X \backslash C^{+}(F)$ is of the first category.
(b) If F is upper s -quasi - continuous, then $X \backslash C^{-}(F)$ is of the first category.

2.4 Corollary.

Let X be a topological space and Y a locally connected regular one. A function $f : X \to Y$ is continuous if and only if it is s -quasi - continuous and almost continuous.

3. Functions of two variables

Let X, Y, Z be topological spaces and $F : X \times Y \to Z$ a multivalued map. For any $x \in X$, $y \in Y$ by F_x, F^y we denote the multivalued maps $F_x : Y \to Z$ and $F^y : X \to Z$ defined by $F_x(y) = F(x, y) = F^y(x)$.

3.1 Theorem.

Let X be a Baire space, Y a locally second countable and let Z be a locally connected regular one. If $F : X \times Y \to Z$ is a multivalued map with compact values such that F_x, F^y are upper s - quasi - continuous for each

$x \in X, y \in Y$ and all the F^y are lower quasi-continuous, then F is upper s - quasi - continuous.

Proof. We apply the method analogous to that in papers [18, 19]. Assume that F is not upper s - quasi - continuous at (x_0, y_0). Then, there exists an open set W containing $F(x_0, y_0)$ and having the connected complement, and there exists a neighbourhood $U \times V$ of (x_0, y_0) such that every non-empty open set $U' \times V' \subset U \times V$ has a point (x, y) for which it holds $F(x', y') \cap (Z \backslash W) \neq \emptyset$. Without loss of generality we can assume that V has a countable base $\{V_n : n \geq 1\}$. The set $M = Z \backslash W$ is closed, $F(x_0, y_0)$ compact, so we can choose disjoint open sets $W', W'' \subset Z$ such that $F(x_0, y_0) \subset W'$ and $M \subset W''$. Since Z is a locally connected space and M a connected subset, there exists an open connected set $W_1 \subset Z$ such that $M \subset W_1 \subset W''$. Hence, $W' \cap Cl(W_1) = \emptyset$ which implies

$$F(x_0, y_0) \subset Z \backslash Cl(W_1) \subset Cl(Z \backslash Cl(W_1)) \subset W.$$

Let $W_0 = Z \backslash Cl(W_1)$; then W_0 is an open set with the connected complement. The map F^{y_0} is upper s - quasi - continuous at x_0, therefore $U_1 = U \cap Int(F^{y_0})^+(W_0)$ is an open non-empty set. Let us put $A_n = \{x \in U_1 : V_n \subset (F_x)^+(W_0)\}$. For $x \in U_1$ we have $F_x(y_0) = F(x, y_0) \subset W_0$. Since F_x is upper s - quasi -continuous, we obtain $Int(F_x)^+(W_0) \cap V \neq \emptyset$, so $V_n \subset (F_x)^+(W_0)$ for some $n \geq 1$. Thus $x \in A_n$ and consequently $U_1 = \bigcup_{n=1}^{\infty} A_n$.

Let $n \geq 1$ be established and let U' be a non-empty open subset of U. Then, for some point $(x_1, y_1) \in U' \times V_n$, it holds that $F(x_1, y_1) \cap (Z \backslash Cl(W_0)) \neq \emptyset$. The lower quasi - continuity of F^{y_1} at x_1 implies the existence of a non - empty open set $U'' \subset U'$ such that

$$F(x, y_1) \cap (Z \backslash Cl(W_0)) \neq \emptyset \text{ for } x \in U''.$$

Hence, $y_1 \notin (F_x)^+(W_0)$ for $x \in U''$. But $y_1 \in V_n$, so $V_n \not\subset (F_x)^+(W_0)$ for $x \in U''$. This means $U'' \cap A_n = \emptyset$ and in consequence U_1 is of the first category. This contradiction finishes the proof.

3.2 Theorem.

Let X be a Baire space, Y a locally second countable and let Z be a locally connected regular one. If $F : X \times Y \to Z$ is a multivalued map such that F_x, F^y are lower s - quasi - continuous for each $x \in X, y \in Y$ and all F^y are upper quasi - continuous, then F is lower s - quasi - continuous.

Proof. Assume that F is not lower s - quasi - continuous at a point (x_0, y_0). Then, there exists an open set $W \subset Z$ with $F(x_0, y_0) \cap W \neq \emptyset$ and $Z \backslash W$ connected, and there exists a neighbourhood $U \times V$ of (x_0, y_0) such that each non-empty open set $U' \times V' \subset U \times V$ contains a point (x', y') for wich $F(x', y') \subset Z \backslash W$. We can assume that V has a count

able base $\{V_n : n \geq 1\}$. Let $z_0 \in F(x_0, y_0) \cap W$. Since $M = Z \backslash M$ is a closed connected set, there exists an open connected set W_1 such that $M \subset W_1$ and $z_0 \in Z \backslash Cl(W_1) \subset Cl(Z \backslash CL(W_1)) \subset W$. Denoting $W_0 = Z \backslash Cl(W_1)$, we have an open set with the connected complement and $z_0 \in W_0$. Since F^{y_0} is lower s - quasi - continuous at x_0, the set $U_1 = U \cap Int(F^{y_0})^-(W_0)$ is non-empty. Let $A_n = \{x \in U_1 : V_n \subset (F_x)^-(W_0)\}$. Using the lower s - quasi - continuity of maps F_x at y_0, we can show the equality $U_1 = \bigcup_{n=1}^{\infty} A_n$. For established $n \geq 1$ and an open non-empty set $U' \subset U_1$, there exists a point $(x_1, y_1) \in U' \times V_n$ such that $F(x_1, y_1) \subset Z \backslash Cl(W_0)$. Since F^{y_1} is upper quasi - continuous at x_1, there exists an open non-empty set $U'' \subset U'$ such that

$$F(x, y_1) \subset Z \backslash Cl(W_0) \text{ for } x \in U''.$$

Thus $y_1 \notin (F_x)^-(W_0)$ for $x \in U''$, and $y_1 \in V_n$, so $V_n \not\subset (F_x)^-(W_0)$ for $x \in U''$. This implies $U'' \cap A_n = \emptyset$. Hence, A_n are nowhere dense sets and U_1 is of the first category.

This is the contradiction finishing the proof.

Let us observe that in the above theorems assumptions on the spaces X, Y, Z are not necessary.

3.3 Example.

Let $Z = [0, \infty)$, $M = \{\frac{1}{n} : n \geq 1\}$ and

$$U_n(x) = \begin{cases} Z \cap (x - \frac{1}{n}, x + \frac{1}{n}), & \text{if } x \neq 0, \\ (0, \frac{1}{n}) \backslash M, & \text{if } x = 0. \end{cases}$$

Then $\{U_n(x) : n \geq 1,\ x \in Z\}$ is a neighbourhood system for some topology T on Z. For each number $a > 0$, the topology T restricted to the subspace $(0, a]$ or $[a, \infty)$ coincides with the natural topology, so

$(0.a]$ and $[a,\infty)$ are connected sets in (Z,T). Moreover, 0 belongs to the T-closure of $(0,a]$, therefore $[0,a]$ is connected. Now, let X be the set of real numbers and $\mathcal{N}$ the σ-ideal of sets of the Lebesgue measure zero. Let us consider the topology $\tau = \{(a,\infty) : a \in X\} \cup \{\emptyset, X\}$; then $\tau_{\mathcal{N}} = \{W \backslash H : W \in \tau, H \in \mathcal{N}\}$ is a topology on X [8].

Assume that $g : (X,\tau) \to (Z,T)$ is an s -quasi - continuous function such that $g(x_1) \leq g(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$. Let us take numbers a_1, a_2 such that $g(x_1) < a_1 < a_2 < g(x_2)$. The sets $[0,a_1)$ and (a_2,∞) are open and have the connected complement in (Z,T) so $Int g^{-1}([0,a_1)) \neq \emptyset \neq Int g^{-1}((a_2,\infty))$. From the definition of τ, for some $b \in X$ we have $(b,\infty) \subset g^{-1}([0,a_1)) \cap g^{-1}((a_2,\infty))$, which is impossible. Hence, only constant functions $g : (X,\tau) \to (Z,T)$ are s - quasi - continuous. It is easy to see that the same holds for functions $g : (X,\tau_{\mathcal{N}}) \to (Z,T)$.

Now, let $f : (X \times X, \tau \times \tau_{\mathcal{N}}) \to (Z,T)$ be a function such that all the f_x are s - quasi - continuous and f^y are quasi - continuous. Then, f_x, f^y are constant functions, so f is too. Thus f is s - quasi - continuous. But (X,T) is not a Baire space, $(X,\tau_{\mathcal{N}})$ is not locally second countable and (Z,T) is not regular [4, p. 61] nor locally connected.

4. Limits of sequences of S - quasi - continuous maps

The pointwise convergence does not preserve the s - quasi - continuity even for real functions.

4.1 Example.

In the space R of real numbers with the natural topology we shall consider functions $f_n, f : R \to R$ given by

$$f_n(x) = \begin{cases} -\frac{1}{n} & \text{for } x \in (-\infty,-1) \\ x^{2n+1} & \text{for } x \in [-1,1] \\ \frac{1}{n} & \text{for } x \in (1,\infty) \end{cases}$$

$$f(x) = \begin{cases} -1 & \text{for } x = -1 \\ 1 & \text{for } x = 1 \\ 0 & \text{for } x \in R \setminus \{-1, 1\}. \end{cases}$$

Then f_n are s - quasi - continuous functions, $f_n \to f$ and f is not s - quasi - continuous at points -1 and 1.

The other situation is in the case of transfinite sequences. To begin with, we shall give some symbols and notions.
Let Ω be the first uncountable ordinal number.

A transfinite sequence $\{a_\xi : \xi < \Omega\}$ of elements of a topological space Y is said to be convergent to $a \in Y$ if for each neighbourhood U of a there exists $\xi < \Omega$ such that $a_\xi \in U$ for each ξ, $\xi_0 \leq \xi < \Omega$, [16]. In the sequel we shall use the following

4.2 Lemma. [16]

Let Y be a first countable T_1-space. If a transfinite sequence $\{a_\xi : \xi < \Omega\}$ converges to a $\in Y$, then, there exists $\xi_0 < \Omega$ such that $a_\xi = a$ for each ξ, $\xi_0 \leq \xi < \Omega$.

A transfinite sequence $\{f_\xi : \xi < \Omega\}$ of maps of X into a topological space Y is called convergent to a map $f : X \to Y$ if for each $x \in X$ the sequence $\{f_\xi(x) : \xi < \Omega\}$ converegs to $f(x)$; then we write $f = \lim_{\xi<\Omega} f_\xi$.

Now, for a transfinite sequence $\{F_\xi : \xi < \Omega\}$ of multivalued maps the symbol $F = \lim_{\xi<\Omega} F_\xi$ means the convergence of all the sequences $\{F_\xi(x) : \xi < \Omega\}$ for $x \in X$, in the space $S(Y)$ of all the non-empty subsets with the Vietoris topology.

4.3 Theorem.

Let X be a locally separable first countable space and Y a topological one such that the space $C(Y)$ of all the non-empty closed subsets of Y is a first countable T_1-space. Assume that $F, F_\xi : X \to Y$, $\xi < \Omega$, are multivalued maps with closed values and $F = \lim_{\xi<\Omega} F_\xi$. If F_ξ are lower (upper) s - quasi - continuous, then F is lower (upper) s - quasi - continuous.

Proof. Suppose that F is not lower (upper) s - quasi - continuous at a point $x_0 \in X$. Then, there exists an open set $V \subset Y$ with $Y \backslash V$ connected and $F(x_0) \cap V \neq \emptyset$ (resp. $F(x_0) \subset V$), and there exists a neighbourhood U of x_0 such that each open non-empty set $U' \subset U$ contains a point x' for which $F(x') \subset Y \backslash V$ (resp. $F(x') \cap (Y \backslash V) \neq \emptyset$). Without loss of generality, let U be separable. Let $\{z_n : n \geq 1\}$ be a dense subset of U and let $\{W_{nj} : j \geq 1\}$ be an open base at z_n, $W_{nj} \subset U$ for $n, j \geq 1$. So, for each $n, j \geq 1$ we can choose a point $x_{nj} \in W_{nj}$ such that

$$(1) \qquad F(x_{nj}) \subset Y \backslash V, \text{ (resp. } F(x_{nj}) \cap (Y \backslash V) \neq \emptyset).$$

According to Lemma 4.2 there exist numbers $\xi_0, \xi_{nj} < \Omega$, $n, j \geq 1$, such that $F_\xi(x_0) = F(x_0)$ for $\xi, \xi_0 \leq \xi < \Omega$,

$$F_\xi(x_{nj}) = F(x_{nj}) \text{ for } n, j \geq 1,\ \xi_{nj} \leq \xi < \Omega.$$

Thus, we can choose $\alpha < \Omega$ such that $\xi_0 < \alpha, \xi_{nj} < \alpha$ for $n, j \geq 1$. Then, we have $F_\alpha(x_0) = F(x_0)$ and $F_\alpha(x_{nj}) = F(x_{nj})$ for $n, j \geq 1$. Since F_α is lower (upper) s - quasi - continuous at x_0, for some W_{nj} it holds

$$F_\alpha(x) \cap V \neq \emptyset \text{ (resp. } F_\alpha(x) \subset V) \text{ for } x \in W_{nj},$$

which is contradictory to (1). Thus the proof is completed.

Now, we are going to consider the uniform limits of s - quasi - continuous maps. For a uniform space $(Y, **)$ by T_{**} is denoted the topology induced by $**$.

For any point $y \in Y$, a set $A \subset Y$ and $W \in **$ we denote by $W[y] = \{x \in Y : (y, x) \in W\}$ and $W[A] = \bigcup\{W[y] : y \in A\}$.

An open cover $\mathcal{A}$ of Y is said to have the Lebesgue property [10] if there exists $W \in **$ such that $\{W[x] : x \in Y\}$ is a refinement.

4.4 Lemma. [10]

Each binary open cover has the Lebesgue property if and only if for any disjoint closed sets $A, B \subset Y$ there exists $W \in **$ such that $W[A] \cap W[B] = \emptyset$.

We shall use the following property
(*) for each connected closed set $M \subset Y$ and each open set U with $M \subset U$, there exists $W \in **$ such that $W[M] \subset U$.

It follows from Lemma 4.4 that if each binary open cover has the Lebesgue property, then (*) is satisfied. Then converse does not hold.

On pages 10-12 instead of the symbol ** to give the script or German U.

4.5 Example.

In the space R with the natural uniformity, the condition (*) is satisfied. But the open cover consisting of the sets

$$U = R\backslash\{1, 2, \ldots\},\ V = \bigcup_{n=1}^{\infty}(n - \frac{1}{n}, n + \frac{1}{n})$$

has not the Lebesque property.

Let $F, F_n, n \geq 1$, be multivalued maps defined on X with values in a uniform space $(Y, **)$. The sequence $\{F_n : n \geq 1\}$ is called uniformly convergent to F, if for each $W \in **$ there is n_0 such that $F_n(x) \subset W[F(x)]$ and $F(x) \subset W[F_n(x)]$ for each $n \geq n_0$, $x \in X$.

4.6 Theorem.

Let X be a topological space, $(Y, **)$ a uniform one with the property (*) and such that (Y, T_{**}) is locally connected. Let $F, F_n : X \to Y, n \geq 1$, be multivalued maps and the sequence $\{F_n : n \geq 1\}$ uniformly converge to F. Then,

(a) If F_n are lower s - quasi - continuous, then F is lower s - quasi - continuous.

(b) If F has the closed values and F_n are upper s - quasi - continuous, then F is upper s - quasi - continuous.

Proof. (a) Let $x_0 \in X, U$ be a neighbourhood of x_0 and let V be an open set with $Y \backslash V$ connected and $F(x_0) \cap V \neq \emptyset$. Let us take $y_0 \in F(x_0) \cap V$, then $y_0 \notin M = Y \backslash V$. Using the regularity of Y and the property (*), we can choose $W \in **$, $W = W^{-1}$ such that $W[y_0] \cap W[M] = \emptyset$. For each $y \in M$ there exists a connected open set V_y satisfying $y \in V_y \subset W[y]$. Because M is connected, the set $V_0 = \bigcup\{V_y : y \in M\}$ is connected and $M \subset V_0 \subset W[M]$. Thus, we have $Cl(V_0) \cap W[y_0] = \emptyset$. Moreover, according to (*) there exists $W_1 \in **, W_1^{-1} = W_1 \subset W$ for which $W_1[M] \subset V_0$. Let n be an established number such that $F_n(x) \subset W_1[F(x)]$ and $F(x) \subset W_1[F_n(x)]$ for each $x \in X$.

Since $y_0 \in F(x_0) \subset W_1[F_n(x_0)]$, for some $z \in F_n(x_0)$ we have $(y_0, z) \in W_1$. From this it follows that $z \notin Cl(V_0)$ and, thus, $F_n(x_0) \cap (Y \backslash Cl(V_0)) \neq \emptyset$. The map F_n is lower s - quasi - continuous at x_0, so there exists an open non-empty set $U_1 \subset U$ such that $F_n(x) \cap (Y \backslash Cl(V_0)) \neq \emptyset$ for $x \in U_1$. Let us take $x \in U_1$ and $y \in F_n(x) \cap (Y \backslash Cl(V_0))$. The condition $y \notin Cl(V_0)$ implies $y \notin W_1[M]$, and $W_1[y] \cap M \neq \emptyset$. From $y \in F_n(x) \subset W_1[F(x)$

] it follows that $(y, z) \in W_1$ for some $z \in F(x)$. Thus $z \notin M$ and consequently, $F(x) \cap (Y \backslash M) \neq \emptyset$. Hence, we have shown $F(x) \cap V \neq \emptyset$ for $x \in U_1$, i.e. F is lower s - quasi - continuous at x_0.

(b) For any $x_0 \in X$ let U be a neighbourhood of this point and $V \subset Y$ an open set with $Y \backslash V$ connected and $F(x_0) \subset V$. The set $M = Y \backslash V$ is connected closed, so applying (*) we can take $W \in **$ such that $W = W^{-1}$ and $W^2[M] \subset Y \backslash F(x_0)$. Hence, we have $W[M] \cap W[F(x_0)] = \emptyset$. Similarly as in the proof of the point (a), we can choose an open connected set V_0 satisfying $M \subset V_0 \subset W[M]$. Thus, $Cl(V_0) \cap W[F(x_0)] = \emptyset$. Using once more the assumption (*), there exists $W_1 \in **, W_1 = W_1^{-1}$ for which $W_1[M] \subset V_0$. Let $n \geq 1$ be an established number such that $F_n(x) \subset W_1[F(x)]$ and $F(x) \subset W_1[F_n(x)]$ for each $x \in X$. Then, we have $F_n(x_0) \subset W_1[F(x_0)] \subset Y \backslash Cl(V_0)$. Since F_n is upper s - quasi - continuous, there exists an open non - empty set $U_1 \subset U$ such that $F_n(x) \subset Y \backslash Cl(V_0)$ for $x \in U_1$. Hence, $F_n(x) \cap W_1[M] = \emptyset$ for $x \in U_1$ and consequently $W_1[F_n(x) \cap M = \emptyset$ for $x \in U_1$. Combining this fact and $F(x) \subset W_1[F_n(x)]$, we obtain $F(x) \subset V$ for $x \in U_1$, which finishes the proof.

5. Remarks on S - quasi - continuous real functions

In this section R denotes the set of real numbers with the natural topology.

A real function f defined on a topological space X is said to be upper (lower) quasi-continuous at a point $x_0 \in X$ if for each neighbourhood U of x_0 and each $\varepsilon > 0$ there exists an open non-empty set $U_1 \subset U$ such that

$$f(x) < f(x_0) + \varepsilon \text{ (resp. } f(x_0) - \varepsilon < f(x)) \text{ for } x \in U_1.$$

A function f is called upper (lower) quasi - continuous if it has this property at each point [6].

The simple examples show that a function which is upper and lower quasi - continuous simultaneously need not be quasi-continuous. But we have

5.1 Remark.

A function $f : X \to R$ is upper and lower quasi-continuous if and only if it is s - quasi - continuous.

We have the following

5.2 Theorem. [6, Th. 6.1 and 6.2]

Let X be a Baire space, Y a second countable one and let $f : X \times Y \to R$ be any function. If f_x, f^y are lower (upper) quasi-continuous for $x \in X, y \in Y$ and f^y are upper (lower) quasi - continuous, then f is lower (upper) quasi - continuous.

Looking carefully at the proofs of these theorems it is easy to see that they are true for a locally second countable space Y. Thus, from the above we obtain immediately.

5.3 Theorem.

Let X be a Baire space, Y a locally second countable one and let $f : X \times Y \to R$ be a function such that f_x, f^y are s - quasi - contionuous for each $x \in X, y \in Y$. Then f is s - quasi - continuous.

Furthermore, we have

5.4 Theorem. [7, Th. 5]

If a function $f : X \to R$ is the pointwise limit of a sequence $\{f_n : n \geq 1\}$ of upper (lower) quasi - continuous functions $f_n : X \to R$, then the set of points at which f is lower (upper) discontinuous is of the first category.

From this it follows simply that:

5.5 Theorem.

If a function $f : X \to R$ is the pointwise limit of a sequence $\{f_n : n \geq 1\}$ of s - quasi - continuous functions $f_n : X \to R$, then the set of points at which f is discontinuous is of the first category.

6. Problems

It is easy to see that for real functions Theorem 6.3 is a stronger result than that which follows from Theorem 3.1 or 3.2. This observation leads to the following

6.1 Problem

Is Theorem 3.1 (resp. 3.2) true if the assumption of lower (upper) quasi - continuity of all the F^y is replaced by lower (upper) s - quasi - continuity?

Let $F_n, F : X \to Y$ be multivalued maps, X, Y topological spaces and let $S(Y)$ be the space of all the non-empty subsets of Y, with the Vietoris topology. We write $F = \lim_{n\to\infty} F_n$, if for each $x \in X$ the sequence $\{F_n(x) : n \geq 1\}$ converges to $F(x)$ in $S(Y)$. We do not know what the limits of upper (lower) s - quasi - continuous maps are. But looking at Theorem 5.5 we obtain

6.2 Problem

Let $F_n, F : X \to Y$ be multivalued maps and $F = \lim_{n\to\infty} F_n$.
If F_n are upper (lower) s - quasi - continuous, then, - perhaps under additional assumptions -is the set of points at which F is not lower (upper) semicontinuous of the first category?

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REZIME

O S – KVAZI – NEPREKIDNIM VIŠEZNAČNIM PRESLIKAVANJIMA

Medju raznim uopštenjima neprekidnosti posmatraju se dva tipa i kod jednog od njih se pretpostavlja da inverzna slika otvorenih skupova pripada jednoj klasi široj nego topologija – ovaj put vodi – specijalno – kvazi – neprekidnim preslikavanjima [11,17]. Kod drugog inverzne slike podskupova neke podfamilije date topologije su razmatrane. Tada, kao primer postoje s – preslikavanja [13,14,15]. Spajanjem obe ove metode klasa s – kvazi – preslikavanja je uredjena.

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