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CONTINUITIES OF MULTI-PRESEMINORMS

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Abstract

By using the results of [8], it is shown that a multi-preseminorm on a product preseminormed space is boundedly uniformly continuous if and only if it is continuous at the origin.

A particular case of this assertion allows the easy proof of some essential improvements of the fundamental theorems of [1, p. 72] and [7] on continuities of multilinear maps.

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1. Prerequisites

Instead of topological vector spaces it is ofter more convenient to use preseminormed spaces [4].

A preseminormed space over $K = R$ or C is a pair $X(\mathcal{P}) = (X, \mathcal{P})$ consisting of a vector space X over K and a nonvoid family $\mathcal{P}$ of preseminorms on X. A preseminorm on X is a subadditive real-valued function p on X such that $p(\lambda x) \leq p(x)$ for all $|\lambda| \leq 1$ and $x \in X$, and $\lim_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in X$. Note that these latter properties imply, in particular, that $p(x) \geq p(0) = 0$

for all $x \in X$, and $p(\lambda x) = p(x)$ for all $|\lambda| = 1$ and $x \in X$.
Among the various basic tools definiable in $X(\mathcal{P})$, we shall actually need only null nets and bounded nets [8].
A net (x_α) in $X(\mathcal{P})$ is a null net if $\lim_\alpha p(x_\alpha) = 0$ for all $P \in \mathcal{P}$. Moreover, a net (x_α) in $X(\mathcal{P})$ converges to a point x in $X(\mathcal{P})$ if $(x_\alpha - x)$ is a null net. And two nets (x_α) and (y_α) in $X(\mathcal{P})$ are coherent [6], if $(x_\alpha - y_\alpha$ is a null net. Note that several useful properties of null nets in $X(\mathcal{P})$ can be easily derived from the usual properties of null nets in R by using the above properties of preseminorms.
A net (x_α) in $X(\mathcal{P})$ is bounded if

$$\lim_{\lambda \to 0} \overline{\lim}_\alpha p(\lambda x_\alpha) = 0$$

for all $p \in \mathcal{P}$. Moreover, a nonvoid subset A of $X(\mathcal{P})$ is bounded if the identity function of A is bounded as a net whenever A is considered to be directed, such that $x \leq y$ for all $x, y \in A$. Note that A is therefore bounded, if and only if

$$\lim_{\lambda \to 0} \sup_{x \in A} p(\lambda x) = 0$$

for all $p \in \mathcal{P}$. And thus nets contained in bounded subsets of $X(\mathcal{P})$ are bounded.
Under this new definition all Cauchy nets in $X(\mathcal{P})$ become bounded. And null nets and bounded nets in $X(\mathcal{P})$ become equivalent tools. Since a net (x_α) in $X(\mathcal{P})$ is bounded if and only if $(\lambda_\beta y_\beta)$ is a null net in $X(\mathcal{P})$, whener (λ_β) is a null net in K and (y_β) is a subnet of (x_α). And a net (x_α) in $X(\mathcal{P})$ is a null net if and only of each subnet (y_β) of (x_α) has a subnet (z_γ) for which there exists an unbounded net (λ_γ) in K such that $(\lambda_\gamma z_\gamma)$ is still a bounded net in $X(\mathcal{P})$. Having the above definition, it is also possible to introduce an important new continuity property of functions.
A function f from a subset D of $X(\mathcal{P})$ into another preseminormed space $Y(\mathcal{Q})$ is boundedly uniformly continuous, if $(f(x_\alpha))$ and $(f(y_\alpha))$ are coherent nets in $Y(\mathcal{Q})$, whenever (x_α) and (y_α) are bounded coherent nets in D. Recall that f may be called uniformly continuous if it takes coherent nets into coherent nets. Thus, if f uniformly continuous, then f is also boundedly uniformly continuous. Moreover, note that if f is boundedly uniformly continuous, then f is necessarily continuous and the restrictions of f to bounded subsets of D are uniformly continuous.
In the sequel, we shall also need a straightforward notion of a product preseminormed space from [5].

If $X_i(\mathcal{P}_i)$ is a preseminormed space for each i in a nonvoid set I, and moreover

$$X = X_{i\in I}X_i \text{ and } \mathcal{P} = \bigcup_{i\in I} \mathcal{P}_i o\pi_i,$$

where π_i is the projection of X onto X_i and $\mathcal{P}_i o\pi_i = \{po\pi_i : p \in \mathcal{P}_i\}$, then the preseminormed space $X(\mathcal{P})$ is the Cartesian product of the spaces $X_i(\mathcal{P}_i)$ and the notation

$$X(\mathcal{P}) = X_{i\in I}X_i(\mathcal{P}_i)$$

is used. An important consequence of this definition is that a net (x_α) in $X(\mathcal{P})$ is a null net (bounded net), if and only if each of its coordinate nets $(x_{\alpha i})$ is a null net (bounded net).

2. Multi-preseminorms

To estabilish easily the continuity properties of multilinear maps, it seems convenient to introduce an appropriate notion of a multi-preseminorm.

Definition 2.1. *Let X_i be a vector space over K for all $i = 1, 2, ..., n$ and*

$$X = X_{i=1}^{n} X_i.$$

For each $x = (x_i) \in X$ and $i = 1, 2, ..., n$, denote by ϕ_{xi} the function defined on X_i by

$$\phi_{xi}(t) = (x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n).$$

A real-valued function p on X will be called a multi- preseminorm on X if
(i) $p_{xi} = po\phi_{xi}$ is a preseminorm on X_i for all $x \in X$ and $i = 1, 2, ..., n$.
(ii) $p_{xi}(\lambda x_i) = p_{xk}(\lambda x_k)$ for all $\lambda \in K, x \in X$ and $i, k = 1, 2, ..., n$.

This rather restrictive definition of a multi-preseminorm is mainly motivated by the next simple example which can easily be extended to multilinear relations.

Example 2.2. If f is a multilinear map of $X = X_{i=1}^{n} X_i$ into a vector space Y and q is a preseminorm on Y, then $p = qof$ is a multi-preseminorm on X.
Another ample reason for having Definition 2.1. is that it allows us to naturally extend several useful properties of preseminorms to multi-preseminorms. First, we shall prove a rather complicated, but important inequality for multi-preseminorms whose one-dimensional particular case is, however, quite trivial.

Lemma 2.3. *If p is a multi-preseminorm on $X = X_{i=1}^{n} X_i$, and moreover $x, y \in X$ and $I = \{1, 2, ..., n\}$, then*

$$p(x) \leq \sum_{A \subset I} p(\chi_A(x-y) + (\chi_I - \chi_A)y),$$

where χ_A is the characteristic function of A on I and the multiplication is taken in the usual pointwise sense.

Proof. By Definition 2.1., it is clear that

$$p(x) \leq p(x_1, ..., x_{n-1}, x_n - y_n) + p(x_1, ..., x_{n-1}, y_n),$$

and the mapping

$$z \epsilon X_{i=1}^{n-1} X_i \to p(z_1, ..., z_{n-1}, s)$$

is a multi-preseminorm on $X_{i=1}^{n-1} X_i$ for all $s \in X_n$.
Thus, if the corresponding inequality holds for $n-1$, then

$$p(x) \leq \sum_{A \subset I \setminus \{n\}} p(\chi_A(x-y) + (\chi_{I \setminus \{n\}} - \chi_A)y + \chi_{\{n\}}(x-y)) +$$

$$+ \sum_{A \subset I \setminus \{n\}} p(\chi_A(x-y) + (\chi_{I \setminus \{n\}} - \chi_A)y + \chi_{\{n\}}y) =$$

$$= \sum_{A \subset I \setminus \{n\}} p(\chi_{A \cup \{n\}}(x-y) + (\chi_I \setminus \chi_{A \cup \{n\}})y) +$$

$$\sum_{A \subset I \setminus \{n\}} p(\chi_A(x-y) + (\chi_I - \chi_A)y)$$

whence the stated inequality follows immediately.

Remark 2.4. Note that this lemma depends only on the multi-subadditivity of p.

Next, we shall prove a very unusual statement about continuities of multi-preseminorms which depends only on the easy consequence of Definition 2.1. that $p(\lambda x) = p(x)$ whenever $\lambda \epsilon K^n$, such that $\prod_{i=1}^n |\lambda_i| = 1$.

Lemma 2.5. *If p is a multi-preseminorm on the product preseminormed space*

$$X(\mathcal{P}) = X_{i=1}^n X_i(\mathcal{P}_i),$$

such that p is continuous at the origin of $X(\mathcal{P})$ and (x_α) is a bounded net in $X(\mathcal{P})$, such that $(x_{\alpha i})$ is a null net in $X_i(\mathcal{P}_i)$ for some $i = 1, 2, ..., n$, then $(p(x_\alpha))$ is a null net in R.

Proof. To prove this, we need only show that each subnet $(p(y_\beta))$ of $(p(x_\alpha))$ has a subnet $(p(z_\gamma))$ such that $(p(z_\gamma))$ is a null net in R. If (y_β) is a subnet of (x_α), then, because of the assumption that $(x_{\alpha i})$ is a null net in $X_i(\mathcal{P}_i)$, it is clear that $(y_{\beta i})$ is also a null net in $X_i(\mathcal{P}_i)$. Thus, by [8, Lemma 3.3], there exist a subnet (z_γ) of (y_β) and a net (λ_γ) of positive numbers with $\lim_\gamma \lambda_\gamma = \infty$ such that $(\lambda_\gamma z_{\gamma i})$ is still a null net in $X_i(\mathcal{P}_i)$. Define $\omega_\gamma \epsilon R^n$ for all γ such that

$$\omega_{\gamma i} = \lambda_\gamma \quad \text{and}$$

$$\omega_{\gamma k} = {}^{n-1}\sqrt{\lambda_\gamma^{-1}} \quad \text{if } k \in \{1, 2, ..., n\} \setminus \{i\}.$$

Then, because of $\prod_{k=1}^n \omega_{\gamma k} = 1$, we clearly have

$$p(z_\gamma) = p(\omega_\gamma z_\gamma)$$

for all γ. Moreover, because of the assumed boundedness of (x_α) and some basic properties of bounded nets listed in Section 1, it is clear that $(\omega_\gamma z_\gamma)$ is a null net in $X(\mathcal{P})$. Thus, because of the assumed continuity of p at o and the above equality, $(p(z_\gamma))$ is a null net in R, and the proof is complete.

Remark 2.6. For an easy illustration of this lemma, one can at once consider the bipreseminorm p defined on R^2 by $p(x) = |x_1||x_2|$.
Combining Lemma 2.3. and Lemma 2.5., we can now easily prove the following useful criterion for the bounded uniform continuity of multi-preseminorms.

Theorem 2.7. *If p is a multi-preseminorm on the product preseminormed space*

$$X(\mathcal{P}) = X_{i=1}^n X_i(\mathcal{P}_i),$$

then the following assertions are equivalent:
(i) p is boundedly uniformly continuous;
(ii) p is continuous at the origin of $X(\mathcal{P})$.

Proof. Since the implication $(i) \Rightarrow (ii)$ is apparent, we need only show that (ii) also implies (i). For this, assume that (ii) holds and (x_α) and (y_α) are bounded coherent nets in $X(\mathcal{P})$. Then, by Lemma 2.3., we clearly have

$$|\, p(x_\alpha) - p(y_\alpha)\,| \leq \sum_{\emptyset \neq A \subset I} (p(\chi_A(x_\alpha - y_\alpha) +$$

$$+(\chi_I - \chi_A)y_\alpha) + p(\chi_A(y_\alpha - x_\alpha) + (\chi_I - \chi_A)x_\alpha))$$

for all α. Hence, by the assumption and Lemma 2.5., it is clear that

$$\lim_\alpha |\, p(x_\alpha) - p(y_\alpha)\,| = 0.$$

That is, $(p(x_\alpha))$ and $(p(y_\alpha))$ are coherent nets in R, and thus (i) also holds.

Remark. To see that this theorem includes Lemma 2.5., note that if (x_α) is as in Lemma 2.5., then by defining $y_\alpha = (\chi_I - \chi_{\{i\}})x_\alpha$ for all α, we get a bounded net (y_α) in $X(\mathcal{P})$ such that (x_α) and (y_α) are coherent nets in $X(\mathcal{P})$ and $p(y_\alpha) = 0$ for all α.

3. An application

By using a convenient reformulation of [1, (18.3) Lemma] and a particular case of Lemma 2.5., we can now easily prove as essential improvement of [1, (18.2) Theorem].

Theorem 3.1. *If f is a multilinear map from a product preseminormed space*

$$X(\mathcal{P}) = X_{i=1}^n X_i(\mathcal{P}_i)$$

into an arbitrary preseminormed space $Y(\mathcal{Q})$, *then the following assertions are equivalent:*
(i) f is boundedly uniformly continuous;
(ii) f is continuous at the origin of $X(\mathcal{P})$;
(iii) $q o f$ *is continuous at the origin of* $X(\mathcal{P})$ *for all* $q \in \mathcal{Q}$.

Proof. It is clear that (i) implies (ii), and (ii) implies (iii).
To prove that (iii) also implies (i), assume that (iii) is true, and let (x_α) and (y_α) be bounded coherent nets in $X(\mathcal{P})$, and moreover $q \in \mathcal{Q}$ and $p = qop$. Then, similarly as in Lemma 2.3., we have

$$f(x_\alpha) - f(y_\alpha) = \sum_{\emptyset \neq A \subset I} f(\chi_A(x_\alpha - y_\alpha) + (\chi_I - \chi_A)y_\alpha),$$

and hence

$$q(f(x_\alpha) - f(y_\alpha)) \leq \sum_{\emptyset \neq A \subset I} p((\chi_A(x_\alpha - y_\alpha) + (\chi_I - \chi_A)y_\alpha)$$

for all α. On the other hand, by Lemma 2.5., it is clear that

$$\lim_\alpha p(\chi_A(x_\alpha - y_\alpha) + (\chi_I - \chi_A)y_\alpha) = 0,$$

whenever $\emptyset \neq A \subset I$. Consequently, we have

$$\lim_\alpha q(f(x_\alpha) - f(y_\alpha)) = 0.$$

And this proves (i).

Remark 3.2. By [8, Theorems 2.2, 2.11 and 2.13] it is clear that a boundedly uniformly continuous map also preserves Cauchy nets.
Thus, the above theorem also improves the important particular case of a recent theorem of Jürg Rätz [3, Theorem 11] when the valuated field is K.
For an easy illustration of this theorem, we can at once state

Theorem 3.3. *If $X(\mathcal{P})$ is a preseminormed space over K, then the scalar multiplication defined on $K \times X$ by*

$$(\lambda, x) \to \lambda x$$

is a boundedly uniformly continuous map of $K(||) \times X(\mathcal{P})$ into $X(\mathcal{P})$ which is uniformly continuous if and only if $\mathcal{P} = \{0\}$.

Proof. If $((\lambda_\alpha, x_\alpha))$ is a null net in $K(||) \times X(\mathcal{P})$ and $p \in \mathcal{P}$, then because of the inequality

$$p(\lambda_\alpha x_\alpha) \leq ([| \lambda_\alpha |] + 1)p(x_\alpha)$$

where [ ] denotes the integer part, it is clear that $(\lambda_\alpha x_\alpha)$ is a null net in $X(\mathcal{P})$. Thus, by Theorem 3.1, the scalar multiplication in $X(\mathcal{P})$ is always

boundedly uniformly continuous.
On the other hand if $\mathcal{P} \neq \{0\}$, then there exist $p \in \mathcal{P}$ and $x \in X$ such that $p(x) \neq 0$. Thus, by defining

$$\lambda_n = n + n^{-1}, \ \mu_n = n \text{ and } x_n = y_n = nx$$

for all positive integers n, we can get coherent sequences $((\lambda_n, x_n))$ and $((\mu_n, y_n))$ in $K(||) \times X(\mathcal{P})$ such that $(\lambda_n x_n)$ and $(\mu_n y_n)$ are not coherent sequences in $X(\mathcal{P})$. Thus, in this case, the scalar multiplication in $X(\mathcal{P})$ is not even sequentially uniformly continuous.
Finally, we note that if $\mathcal{P} = \{0\}$, then any two nets in $X(\mathcal{P})$ are coherent, and thus the uniform continuity of the scalar multiplication in $X(\mathcal{P})$ holds in a trivial manner.

Remark 3.4. Note that the addition in $X(\mathcal{P})$ and the members of $\mathcal{P}$ are always uniformly continuous.

References

[1] S.K.Berberian, Lecture in Functional Analysis and Operator Theory, Springer-Verlag, Berlin 1974.

[2] J.L.Kelley, General Topology, Van Nostrand Reinhold Company, New York, 1955.

[3] J.Rätz, On continuous multilinear mappings, C.R.Math.Rep. Acad.Sci. Canada 8(1986), 219-224.

[4] Á.Száz, Preseminormed spaces, Publ.Math. Debrecen 30(1983), 217-224.

[5] Á.Száz, Projective generation of preseminormed spaces by linear relations, Studia Sci. Math. Hungar., 23(1988), 297-313.

[6] Á.Száz, Coherences instead of convergences, Proceedings of the Conference on Convergence and Generalized Functions, Katowice 1983, Institute of Mathematics, Polish Academy of Sciences, Warsaw 1984, pp. 141-148.

[7] Á.Száz, Nets of multilinear maps, Abstracts and Program of the Conference on Generalized Functions, Convergence Structures and their Applications, Dubrovnik 1987, Institute of Mathematics, Novi Sad 1987, p. 55.

[8] Á.Száz, Bounded nets in preseminormed spaces, Period. Math. Hungar., to appear.

REZIME

NEPREKIDNOST MULTI – PRESEMINORMI

Korišćenjem rezultata iz [8], pokazano je da multipreseminorma nad proizvod preseminormiranom prostoru je ograničeno uniformno neprekidna ako i samo ako je neprekidna u nuli.

Specijalan slučaj ovog tvrdjenja omogućava jednostavan dokaz nekih bitnih poboljšanja fundamentalnih teorema iz [1,p.72] i [7] o neprekidnosti multilinearnih preslikavanja.

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