Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 1 (1990), 135-144

Review of Research
Faculty of Science
Mathematics Series

# AN INTEGRAL GENERATED BY A DECOMPOSABLE MEASURE

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**Abstract**

An integral using the pseudo-addition $\oplus$, pseudo-multiplication $\otimes$ and the $\bigoplus$−decomposable measure is introduced. The method used is similar to the procedure of the construction of Lebesgue integral.

*AMS Mathematics Subject Classification (1980):* 28A25
*Key words and phrases:* Pseudo-addition, pseudo-multiplication, decomposable measure.

## 1. Introduction

There are many different ways for defining integrals with respect to non-additive set functions (some of them in [1], [2], [3], [4], [7], [8], [9]). We continue our investigations of $\bigoplus$−decomposable measures initiated in paper [5] based on the pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ defined on $[a, b] \subset [-\infty, +\infty]$. In this paper we shall examine the corresponding integral using a construction similar to that of the Lebesgue integral.

## 2. Decomposable measures

Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,+\infty]$. We shall consider a partial order $\leq$ on $[a,b]$, which can be the usual order of the real line, but it can also be another order. All future considerations will be with respect to the order $\leq$ .

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b]\times[a,b] \to [a,b]$ which is commutative, nondecreasing (with respect to $\leq$) associative and either $a$ or $b$ is a zero element, denoted by 0, i.e. for each $x \in [a,b]$

$$0 \oplus x = x \quad \text{holds.}$$

Let $[a,b]_+ = \{x : x \in [a,b],\ \ x \geq 0\}$.

The operation $\otimes$ (pseudo-multiplication) is a function $\otimes : [a,b]\times[a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leq y$ implies $x \otimes z \leq y \otimes z$, $z \in [a,b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a,b]$, i.e. for each $x \in [a,b]$

$$\mathbf{1} \otimes x = x.$$

We suppose, further, $0 \otimes x = 0$ and that $\otimes$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z).$$

Examples can be found in paper [5]. Some of them are:

$$x \oplus y = (x^p + y^p)^{\frac{1}{p}},\ p > 0 \quad \text{and} \quad x \otimes y = x \cdot y \ \text{ on } [0,+\infty]$$

$$\text{or} \quad x \oplus y = \max\{x,y\} \quad \text{and} \quad x \otimes y = x + y \ \text{ on } [-\infty,+\infty).$$

Pseudo-addition $\oplus$ is idempotent if for any $x \in [a,b]$

$$x \oplus x = x \quad \text{holds.}$$

Let $X$ be a non-empty set. Let $\sum$ be a $\sigma-$algebra of subsets of $X$.

A set function $m : \sum \to [a,b]_+$ (or semiclosed interval) is a $\oplus-$decomposable measure if there hold

$$m(\emptyset) = 0 \quad (\text{if } \oplus \text{ is not idempotent})$$

$$m(A \cup B) = m(A) \oplus m(B)$$

for $A, B \in \sum$ such that $A \cap B = \emptyset$.
In the case when $\oplus$ is idempotent, it is possible that $m$ is not defined on the empty set.

A $\bigoplus$−decomposable measure $m$ is $\sigma$ - $\bigoplus$−decomposable if

$$m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence $(A_i)$ of pairwise disjoint sets from $\sum$.

**Proposition 1.** *If $m : \sum \to [a, b]_+$ is a $\bigoplus$−decomposable measure with respect to the idempotent pseudo-addition $\oplus$, then we have*

$$m(A \cup B) = m(A) \oplus m(B)$$

*for any $A, B \in \sum$.*

## 3. Integral

Let $m$ be a $\sigma$ - $\bigoplus$−decomposable measure. A function $f : X \to [a, b]$ is measurable from below if for any $c \in [a, b]$ the sets $\{x : f(x) \leq c\}$ and $\{x : f(x) < c\}$ belong to $\sum$. $f$ is measurable, if it is measurable from below and the sets $\{x : f(x) \geq c\}$ and $\{x : f(x) > c\}$ belong to $\sum$.

Let $f$ and $g$ be two functions defined on $X$ and with values in $[a, b]$. Then, we define for any $x \in X$

$$(f \oplus g)(x) = f(x) \oplus g(x) \ ,$$

$$(f \otimes g)(x) = f(x) \otimes g(x)$$

and for any $c \in [a, b]$

$$(c \otimes f)(x) = c \otimes f(x).$$

We suppose further that $([a,b],\oplus)$ and $([a,b],\otimes)$ are complete lattice ordered semigroups. A complete lattice means that for each set $A \subset [a,b]$ bounded from above (below) there exists $\sup A$ ($\inf A$). Further, we suppose that $[a,b]$ is endowed with a metric $d$ compatible with sup and inf and which satisfies at laest one of the following conditions:

(a) $d(x \oplus y,\ x' \oplus y') \leq d(x,x') + d(y,y')$

(b) $d(x \oplus y,\ x' \oplus y') \leq \max\{d(x,x'),\ d(y,y')\}$.

Both conditions $(a)$ and $(b)$ imply that :

$$d(x_n, y_n) \to 0 \quad \text{implies} \quad d(x_n \oplus z,\ y_n \oplus z) \to 0.$$

Condition $(b)$ implies

$$d(\bigoplus_{i=1}^{n} x_i, \bigoplus_{j=1}^{n} y_j) \leq \min_j \max_i d(x_i, y_j).$$

We suppose further the monotonicity of the metric $d$, i.e.

$$x \leq z \leq y \quad \text{implies} \quad d(x,y) \geq \max\{d(y,z), d(x,z)\}.$$

Let $\varepsilon$ be a positive real number, and $B \subset [a,b]$. A subset $\{l_i^\varepsilon\}$ is a $\varepsilon-$net if for each $x \in B$ there exists $l_i^\varepsilon$ such that $d(l_i^\varepsilon, x) \leq \varepsilon$. If we have $l_i^\varepsilon \leq x$, then we shall call $\{l_i^\varepsilon\}$ a lower $\varepsilon-$net. If $l_i^\varepsilon \leq l_{i+1}^\varepsilon$ holds, then $\{l_i^\varepsilon\}$ is monotone.

We define the characteristic function

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}.$$

A mapping $e : X \to [a,b]$ is an elementary (measurable) function if it has the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \quad \text{for} \quad a_i \in [a,b]$$

and $A_i \in \sum$ disjoint if $\oplus$ is not idempotent.

**Theorem 1.** *Let $f : X \to [a,b]$ be a measurable from below function if the pseudo-addition is idempotent, or $f$ is measurable and for each positive real number $\varepsilon$ there exists a monotone $\varepsilon$-net in $f(X)$. Then, there exist a sequence $(\varphi_n)$ of elementary functions such that, for each $x \in X$,*

$$d(\varphi_n(x), f(x)) \to 0 \quad \textit{uniformly as} \quad n \to \infty.$$

*Proof.* Suppose first that $\oplus$ is not idempotent.We take a lower monotone $\varepsilon$-net $\{f_i^\varepsilon\}$ on $f(X)$.

Let

$$e^\varepsilon = \bigoplus_{i=1}^{\infty} f_i^\varepsilon \otimes \chi_{X_i^\varepsilon},$$

where $X_i^\varepsilon = \{x : f_{i+1}^\varepsilon > f(x) \geq f_i^\varepsilon\}$.
For each point $x$ from $X$ there exists $f_i^\varepsilon(x)$ such that

$$d(f_i^\varepsilon(x), f(x)) \leq \varepsilon,$$

where $f_i^\varepsilon(x) = f_i^\varepsilon \otimes \chi_{X_i^\varepsilon}(x)$. Hence, by the monotonicity of $d$ and $e^\varepsilon(x) \geq f_i^\varepsilon(x)$, we obtain

$$d(e^\varepsilon(x), f(x)) < \varepsilon, \quad x \in X.$$

Taking $\varepsilon = \frac{1}{n}$ we define the desired sequence $(\varphi_n)$ as $\varphi_n = e^{\frac{1}{n}}$.

If the pseudo-addition $\oplus$ is idempotent, then we take in the preceding procedure a lower $\varepsilon$-net $\{f_i^\varepsilon\}$ and

$$X_i^\varepsilon = \{x : f(x) \geq f_i^\varepsilon\}.$$

We have used that $e^\varepsilon(x) \leq f(x)$ holds, which follows by

$$e^\varepsilon(x) \leq e^\varepsilon(x) \oplus f(x) \leq \bigoplus_{i=1}^{\infty}((f_i^\varepsilon \otimes \chi_{X_i^\varepsilon}(x)) \oplus f(x)) \leq f(x).$$

The integral of a simple function $s = \bigoplus_{i=1}^{n} a_i \otimes \chi_{A_i}$ for $a_i \in [a,b]$ with disjoint $A_1, A_2, \ldots A_n$, if $\oplus$ is not idempotent, is defined by

$$\int_X^{\oplus} s \otimes dm := \bigoplus_{i=1}^{n} a_i \otimes m(A_i).$$

The integral of an elementary function

$$e = \bigoplus_{i=1}^{\infty} a_i \otimes \chi_{A_i} \quad \text{for} \quad a_i \in [a,b] \ (i \in N) \ \text{with} \ (A_i)$$

disjoint if $\oplus$ is not idempotent, is defined by

$$\int_X^{\oplus} e \otimes dm := \bigoplus_{i=1}^{\infty} a_i \otimes m(A_i). \tag{1}$$

The integral of a bounded measurable (from below for $\oplus$ idempotent) function $f : X \to [a,b]$, for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$, there exists a monotone $\varepsilon$−net in $f(X)$, is defined by

$$\int_X^{\oplus} f \otimes dm := \lim_{n\to\infty} \int_X^{\oplus} \varphi_n(x) \otimes dm, \tag{2}$$

where $(\varphi_n)$ is the sequence of elementary functions constructed in Theorem 1.

**Theorem 2..** *The integral defined in (2) is independent of the choice of sequence* $(\varphi_n)$.

**Theorem 3.** *Let $\oplus$ and $\otimes$ be continuous and $\oplus$ infinitely commutative and associative. Then the integrals defined by (1) and (2) have the following properties:*

*(i)* $\int_X^{\oplus} (f \oplus g) \otimes dm = \int_X^{\oplus} f \otimes dm \oplus \int_X^{\oplus} g \otimes dm$ ,

*(ii)* $\int_X^{\oplus} (c \otimes f) \otimes dm = c \otimes \int_X^{\oplus} f \otimes dm$

*for any* $c \in [a,b]$.

*Proof.* $(i)$ Let $f$ and $g$ be elementary functions, i.e.

$$f = \bigoplus_{i=1}^{\infty} (a_i \otimes \chi_{A_i}), \quad g = \bigoplus_{i=1}^{\infty} b_i \otimes \chi_{B_i},$$

where $(A_i)$ and $(B_i)$ are (if $\oplus$ is not idempotent, disjoint) partitions of $X$ in $\sum$. Hence, $f \oplus g$ is also an elementary function and

$$f \oplus g = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (a_i \oplus b_j) \otimes \chi_{A_i \cap B_j}.$$

By (1) we obtain

$$\int_X^{\oplus} (f \oplus g) \otimes dm = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (a_i \oplus b_j) \otimes m(A_i \cap B_j) =$$

$$= \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} (a_i \otimes m(A_i \cap B_j)) \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} b_j \otimes m(A_i \cap B_j) =$$

$$= \bigoplus_{i=1}^{\infty} (a_i \otimes m(A_i \cap \cup_{j=1}^{\infty} B_j)) \oplus \bigoplus_{j=1}^{\infty} (b_j \otimes m(\cup_{i=1}^{\infty} A_i \cap B_j) =$$

$$= \bigoplus_{i=1}^{\infty} (a_i \otimes m(A_i)) \oplus \bigoplus_{j=1}^{\infty} (b_j \otimes m(B_j)) =$$

$$= \int_X^{\oplus} f \otimes dm \oplus \int_X^{\oplus} g \otimes dm$$

Now, let $f$ and $g$ be measurable (from below if $\oplus$ is idempotent). Let $(\varphi_n)$ and $(\psi_n)$ be the corresponding sequences from Theorem 1 to $f$ and $g$, respectively. The integral

$$\int_X^{\oplus} (f \oplus g) \otimes dm$$

exists, since it can be defined by the sequence $(\varphi_n(x) \oplus \psi_n(x))$

$$\int_X^{\oplus} (f \oplus g) \otimes dm = \lim_{n \to \infty} \int_X^{\oplus} (\varphi_n(x) \oplus \psi_n(x)) \otimes dm$$

and $(\varphi_n(x))$ and $(\psi_n(x))$ satisfy, for any $x$,

$$d(\varphi_n(x), f(x)) \to 0 \quad \text{and} \quad d(\psi_n(x), g(x)) \to 0.$$

Hence, since $d$ satisfies $(a)$ or $(b)$

$$d(\varphi_n(x) \oplus \psi_n(x), f(x) \oplus g(x)) \to 0.$$

Now, we have

$$\int_X^{\oplus}(f \oplus g) \otimes dm = \lim_{n\to\infty} \int_X^{\oplus} (\varphi_n \oplus \psi_n) \otimes dm =$$

$$= \lim_{n\to\infty} (\int_X^{\oplus} \varphi_n \otimes dm \oplus \int_X^{\oplus} \psi_n \otimes dm) =$$

$$= \lim_{n\to\infty} \int_X^{\oplus} \varphi_n \otimes dm \oplus \lim_{n\to\infty} \int_X^{\oplus} \psi_n \otimes dm =$$

$$= \int_X^{\oplus} f \otimes dm \oplus \int_X^{\oplus} g \otimes dm.$$

Property $(ii)$ easily follows by the continuity of $\otimes$.

**Example 1.** *For any function $g$ bounded above we can define*

$$m(A) = \sup_{x\in A} g(x) \qquad A \in \mathcal{B},$$

*where $\mathcal{B}$ is the Borel $\sigma-$algebra on $[-\infty, \infty)$.*
*Taking $\oplus = \max = \sup$, $\otimes = +$ , we obtain*

$$\int_{\mathbf{R}}^{\oplus} f \otimes dm = \sup_{x\in\mathbf{R}} (f(x) + g(x)),$$

*$f$ bounded above.*

*If $\oplus$ is a strict pseudo-addition with a monotone generator $g$, $g \circ m : \sum \to [o, g(c)]$ and $c \in [a,b]$ is an additive measure then we have (see* [3], [5]*) for the simple function*

$$\int_X^{\oplus} s \otimes dm = g^{-1}(\sum_{i=1}^{n} g(a_i) \cdot (g \circ m)(A_i))$$

*and for the measurable function $f$*

$$\int_X^{\oplus} f \otimes dm = g^{-1}(\int_X (g \circ f) \cdot dx),$$

*where $dx = d(g \circ m)$ is the Lebesgue measure and $u \otimes v = g^{-1}(g(u) \cdot g(v))$.*

**Example 2.** ([5]) *If $c > 0$, then we define*

$$u \oplus v = -c\ln(e^{-\frac{u}{c}} + e^{-\frac{v}{c}}) \quad and$$

$$u \otimes v = u + v.$$

*The corresponding integral is*

$$\int_R^{\oplus} f \otimes dm = -c\ln \int_{\mathbf{R}} e^{(-\frac{f}{c})} dx.$$

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**REZIME**

INTEGRAL GENERISAN DEKOMPOZABILNOM MEROM

Uveden je integral pomoću pseudo-sabiranja $\oplus$, pseudo-množenja $\otimes$ i $\bigoplus$ -- dekompozabilne mere. Korišćena je metoda bliska konstrukciji Lebesgueovog integrala.

*Received by the editors November 15, 1990.*