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A NOTE ON ITERATIVE PROCESS FOR SOLVING LINEAR SYSTEMS OF EQUATIONS

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Abstract

A simple modification of an algorithm from [3] for numerical solution of linear system of equations is considered. The numerical examples show that introducing a parameter the basic method can be accelerated.

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1. Introduction

Research in the area of economic models is concentrated on determining the convergence conditions for some solution tehniques applied to systems of equations which have none of the usual features of physical systems, such as symmetry, diagonal dominance, or nonnegativity, Fisher, Hallet, [1]. In Kudrinskii, Ostapchuk, [3], an algorithm for the iterative solution of the system $Ax=b$, assuming only that matrix A is nonsingular, is given. This algorithm is nonstationary and can be applied without the user's interventions. Some numerical experiments show that by introducing a parameter, the basic method can be accelerated.

In this paper we shall prove the convergence of our modification of the basic method assuming that the introduced parameter satisfies a condition.

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2. Preliminaries

We shall consider a system of linear equations

$$Ax = b$$

where $A = [a_{ij}] \in \mathbf{R}^n$ is a nonsingular matrix and $b = [b_1, \ldots, b_n]^T$ is a known vector.

Before describing our modification of the basic method from Kudrinskii, Ostapchuk, [3], we shall give some notations.

Let x^T be the transpose of vector $x \in \mathbf{R}^n$ and let $(x, y) = x^T y$ be the inner product of $x, y \in \mathbf{R}^n$. We denote by $\|x\|$ the Euclidean norm: $\|x\| = \sqrt{(x^T, x)}$ and by $\|x\|_\infty$ and $\|A\|_\infty$ the maximum vector norm $(\|x\|_\infty = max\{| x_i | : i = 1, 2, \ldots, n\})$ and the corresponding matrix norm. Let $a_i = [a_{i1}, \ldots, a_{in}], i = 1, 2, \ldots, n$ be row vectors.

We shall use the following notations (for $A \in \mathbf{R}^{n,n}$ and $x \in \mathbf{R}^n$):

$$N = \{1, 2, \ldots, n\}.$$

$$r(x^k) = b - Ax^k = [r_1^k, r_2^k, \ldots, r_n^k]^T \in \mathbf{R}^n.$$

Now, we are going to describe our method. With arbitrary $x^0 \in \mathbf{R}^n$ we form the iterations by

$$x^{k+1} = Gx^k, \quad k = 0, 1, \ldots, \tag{1}$$

where

$$Gx = x + \frac{\sigma}{\|a_i^T\|^2} r_i a_i^T, \tag{2}$$

$$| r_i | = \|r(x)\|_\infty \quad \text{and} \tag{3}$$

$$0 < \sigma < 2. \tag{4}$$

3. Convergence theorem

The proof of the following theorem is based on Kudrinskii, Ostapchuk, [3], and More [4].

Theorem 1. *Let A be a nonsingular matrix. Then for any $x^0 \in \mathbf{R}^n$ the iterations (1)-(4) converge to the unique solution x^* of the system $Ax = b$.*

Proof. The nonsingularity of A ensures that the sequence $\{x^k\}$ is well defined and that the unique solution x^* of the system $Ax = b$ exists. It is easy to see that is continuous for $x \in \mathbf{R}^n$. Let us define

$$\sigma_k = \frac{\sigma}{\|a_{i_k}^T\|^2}.$$

From (1)-(3) we obtain

$$x^{k+1} = x^k + \frac{\sigma}{\|a_{i_k}^T\|^2}\, r_{i_k}^k\, a_{i_k}^T, \quad | r_{i_k}^k | = \|r(x^k)\|_\infty,$$

and

$$x^* - x^{k+1} = x^* - x^k - \sigma_k\, r_{i_k}^k\, a_{i_k}^T,$$

$$\|x^* - x^{k+1}\|^2 = (x^* - x^k - \sigma_k\, r_{i_k}^k\, a_{i_k}^T)^T (x^* - x^k - \sigma_k\, r_{i_k}^k\, a_{i_k}^T)$$

$$= \|x^* - x^k\|^2 - 2\sigma_k\, r_{i_k}^k\, a_{i_k}(x^* - x^k) + (\sigma_k\, r_{i_k}^k\, \|a_{i_k}^T\|)^2.$$

From the definition of $r(x^k)$ we have $r(x^k) = b - Ax^k = Ax^* - Ax^k$ and, thus

$$r_{i_k}^k = a_{i_k}\,(x^* - x^k).$$

So, we now have

$$\|x^* - x^{k+1}\|^2 = \|x^* - x^k\|^2 - \sigma_k\,(r_{i_k}^k)^2\,(2 - \sigma_k \|\, a_{i_k}^T\,\|^2).$$

If $r^k_{i_k} = 0$, then, (3), $\|Ax^k - b\|_\infty = 0$ and follows immediately that x^k is the solution of the system $Ax = b$, and the Theorem is proved.
If $r^k_{i_k} \neq 0$, then because of (4) : $0 < \sigma_k$, $2 - \sigma_k \|a^T_{i_k}\|^2 > 0$, we obtain

$$\|x^* - x^{k+1}\| < \|x^* - x^k\|.$$

So, $\varepsilon_k = \|x^* - x^k\|$ is a decreasing sequence of nonnegative numbers and hence convergent. Thus $\{x^k\}$ is bounded, and if $\{x^{k_i}\}$ is any convergent subsequence such that

$$\lim_{i\to\infty} x^{k_i} = y^* \neq x^*,$$

then

$$\lim_{i\to\infty} \varepsilon_{k_i+1} = \|Gy^* - x^*\| < \|y^* - x^*\| = \lim_{i\to\infty} \varepsilon_{k_i},$$

which is in contradiction with the fact that $\{\varepsilon_k\}$ is convergent. Therefore,

$$\lim_{i\to\infty} x^{k_i} = x^*,$$

and consequently, $\lim_{k\to\infty} x^k = x^*$. □

In Kudrinskii, Ostapchuk, [3], $\sigma = 1$ in each iteration. Now, we are going to consider the following modification of method (1)-(4):

(5) $$x^{k+1} = G_k\, x^k, \quad k = 0, 1, \ldots,$$

(6) $$G_k x = x + \frac{f(k)}{\|a^T_i\|^2}\, r_i a^T_i,$$

(7) $$|\, r_i \,| = \|r(x)\|_\infty \quad \text{and}$$

(8) $$f(k) \in (0,2), \lim_{k\to\infty} f(k) = \sigma \in (0,2).$$

Theorem 2. *Let A be a nonsingular matrix. Then for any $x^0 \in \mathbf{R}$ the iterations (5)-(8) converge to the unique solution x^* of the system $Ax = b$.*

Proof. Let us denote $\sigma_k = \frac{f(k)}{\|a_{i_k}^T\|^2}$. From (5)-(7) we obtain

$$x^{k+1} = x^k + \frac{f(k)}{\|a_{i_k}^T\|^2} r_{i_k}^k a_{i_k}^T , \quad | r_{i_k}^k | = \|r(x^k)\|_\infty .$$

As in the proof of the Theorem 1 we obtain

$$\|x^* - x^{k+1}\| < \|x^* - x^k\|.$$

So, $\varepsilon_k = \|x^* - x^k\|$ is a decreasing sequence of nonnegative numbers and hence convergent. Thus $\{x^k\}$ is bounded and, since

$$G_k x^k - Gx^k = (f(k) - \sigma) \frac{1}{\|a_{i_k}^T\|^2} r_{i_k}^k a_{i_k}^T ,$$

we have for arbitrary small $\varepsilon > 0$

$$\|G_k x^k - Gx^k\| \leq | f(k) - \sigma | \frac{\|r(x^k)\|_\infty}{\|a_{i_k}^T\|} < \varepsilon \text{ for } k > K(\varepsilon). \tag{9}$$

If $\{x^{k_i}\}$ is any convergent subsequence such that

$$\lim_{i\to\infty} x^{k_i} = y^* \neq x^*,$$

then for arbitrary $\varepsilon > 0$ from (9) and the continuity of G it follows that

$$\|G_{k_i} x^{k_i} - Gy^*\| \leq \|G_{k_i} x^{k_i} - Gx^{k_i}\| + \|Gx^{k_i} - Gy^*\| < \varepsilon \text{ for } i > I(\varepsilon),$$

i.e.

$$\lim_{i\to\infty} G_{k_i} x^{k_i} = Gy^*.$$

Now we have

$$\lim_{i\to\infty} \varepsilon_{k_i+1} = \|G_{k_i} x^{k_i} - x^*\| = \|Gy^* - x^*\| < \|y^* - x^*\| = \lim_{i\to\infty} \varepsilon_{k_i},$$

which is in contradiction with the fact that $\{\varepsilon_k\}$ is convergent. Therefore

$$\lim_{i\to\infty} x^{k_i} = x^*,$$

and consequently, $\lim_{i\to\infty} x^k = x^*$. □

4. Numerical examples

In this section we shall describe the results of numerical experiments which were designed to illustrate the effectiveness of the method discussed in the previous section. To do this we solve the test problem $Ax = b$, where

$$A = \begin{bmatrix} 3 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3-1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 3 \end{bmatrix} \in \mathbf{R}^{10,10}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 2 \end{bmatrix} \in \mathbf{R}^{10}.$$

For this problem the solution $\bar{x}$ is known and is unity everywhere. We define the iteration error vector by

$$\varepsilon^k = x^k - \bar{x},$$

and always let E_T denote the true error measure

$$E_T = \frac{\|\varepsilon^k\|}{\|\bar{x}\|}.$$

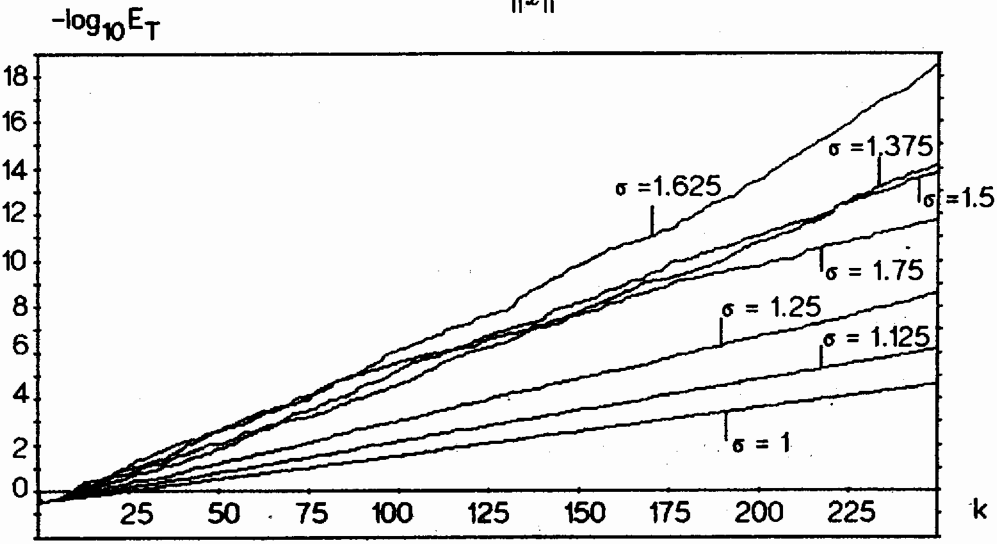

Figure 1.

In Fig. 1 graphs of $-\log_{10} E_T$ versus k are given for different values of σ in solving the problem $Ax = b$. We used the null vector as the initial guess for $\bar{x}$.

Table 1 gives for each σ the number of iterations needed to reduce the norm of the error vector by a factor 10^{-3}, as compared with the norm of the initial error vector, i.e., when $\|\varepsilon^k\|/\|\varepsilon^0\| \leq 10^{-3}$.

In order to illustrate the behaviour of the adaptive procedure (1)-(3) with $x^0 = 0$,

$$(10)\quad f(1) = 1.999,\ f(k) = 2 - \omega + \frac{\omega}{ln(1+k)},\ \omega \in (0,2),\ k = 2,3,\ldots,$$

as a function of ω, we are giving again for each ω the number of iterations needed to have $\|\varepsilon^k\|/\|\varepsilon^0\| \leq 10^{-3}$. As a start value we use $f(0) = 1.999$ for each ω.

Table 1.

σ	k
1.000	293
1.125	226
1.250	170
1.375	112
1.500	104
1.625	94
1.750	99
1.875	192

Table 2.

ω	k
0.2500	141
0.3125	97
0.3750	93
0.4375	86
0.5000	73
0.5625	80
0.6250	83
0.6875	81

Numerical examples show that, very often, our methods (1)-(3) with parametar $\sigma \in (0,2)$, and its adaptive version given by (5)-(8), converge faster than the basic method from [3], i.e. method (1)-(3) with $\sigma = 1$.

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REZIME

NOTA NA ITERATIVNI PROCES ZA REŠAVANJE LINEARNOG SISTEMA JEDNAČINA

Posmatra se jedna modifikacija poznatog postupka za rešavanje sistema linearnih jednačina. Numerički primeri pokazuju da se uvodjenjem parametra osnovni postupak može ubrzati.

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