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ON A CLASS OF BISEMILATTICES

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Abstract. In the papers [1] and [2] a near-lattice was defined as a bisemilattice (Q,∇,Δ) satisfying the identity:

$$x\Delta(y\nabla z\nabla x) = (x\Delta y)\nabla(x\Delta z)\nabla(x\Delta x).$$

This structure is called here a (Δ,∇)-weak-distributive bisemilattice, and the structure satisfying the dual identity is said here to be a (∇,Δ)-weak-distributive bisemilattice.

In this paper a near-lattice is defined as a bisemilattice which satisfies the identity

$$x\nabla(y\Delta x) = (x\nabla y)\Delta x.$$

Some properties of such structures are proved, and a necessary and sufficient condition for a bisemilattice to be a near-lattice is given.

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A bisemilattice (Q,∇,Δ) is an algebra with two binary operations such that (Q,∇) and (Q,Δ) are commutative semigroups which satisfy the idempotent laws.

We shall say that a bisemilattice (Q,∇,Δ) is a near-lattice iff for all $x,y\in Q$

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(SM) $x\nabla(y\Delta x) = (x\nabla y)\Delta x$.

(SM) is a self- dual law, and if (Q,∇,Δ) is a near-lattice, then (Q,Δ,∇) is also a near-lattice. Thus, the duality is satisfied in any near-lattice.

Example 1.

$Q = \{a,b,c\}$

∇	a	b	c
a	a	b	a
b	b	b	b
c	a	b	c

Δ	a	b	c
a	b	b	b
b	b	b	b
c	b	b	c

The bisemilattice (Q,∇,Δ) is not a near-lattice, since $a\nabla(c\Delta a) = b \neq a = (a\nabla c)\Delta a$.

In a bisemilattice (Q,∇,Δ) (SM) follows from the identity $x\Delta(y\nabla z\nabla x) = (x\Delta y)\nabla(x\Delta z)\nabla(x\Delta x)$ (replacing z by x). It also follows from the dual identity. Hence:

Proposition 1. [1][2] Every (Δ,∇)-weak-distributive bisemilattice (as a (∇,Δ)-weak-distributive bisemilattice) is a near-lattice. ■

Example 2.

$Q = \{a,b,c\}$

∇	a	b	c
a	a	b	c
b	b	b	b
c	c	b	c

Δ	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

The bisemilattice (Q,∇,Δ) is a (Δ,∇) -weak-distributive bisemilattice, and so it is a near-lattice, but since $c\nabla(a\Delta b\Delta c)=c \neq b=(c\nabla a)\Delta(c\nabla b)\Delta c$, it is not a (∇,Δ)-weak-distributive bisemilattice.

If (Q,∇,Δ) is a near-lattice, we define partial orders on the set Q by:

$a \leq_\nabla b$ iff $a\nabla b = b$ and

$a \leq_\Delta b$ iff $a\Delta b = a$.

We represent a near-lattice (Q,∇,Δ) by Hasse diagrams of the semilattices (Q,∇) and (Q,Δ). If $a \leq_\nabla b$, then in a Hasse diagram of (Q,∇), we draw b above a, and if $a \leq_\Delta b$, then we do the same in a Hasse diagram of (Q,Δ).

[2] *This proposition gave the initial idea for this paper.*

Example 3.

$Q = \{a,b,c,d,e\}$

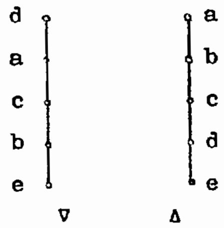


Fig. 1

Since $b\Delta(c\nabla a\nabla b) \neq (b\Delta c)\nabla(b\Delta a)\nabla b$, and $c\nabla(e\Delta d\Delta c) \neq (c\nabla e)\Delta(c\nabla d)\Delta c$, the bisemilattice (Q,∇,Δ) is neither a (∇,Δ), nor a (Δ,∇)-weak -distributive bisemilattice, but it is a near-lattice.

In a near-lattice (Q,∇,Δ), $a\nabla b = b$ implies

$$a\nabla(a\Delta b) = a\Delta(a\nabla b) = a\Delta b, \text{ and } b\nabla(a\Delta b) = (b\nabla a)\Delta b = b.$$

Thus, we have:

Lemma 2. Let (Q,∇,Δ) be a near-lattice. If for $a,b\in Q$, $a \leq_\nabla b$, then $a\nabla(a\Delta b) = a\Delta b$ and $b\nabla(a\Delta b) = b$. ∎

Dually, we have:

Lemma 2'. Let (Q,∇,Δ) be a near-lattice. If for $a,b\in Q$, $b \leq_\Delta a$, then $a\Delta(a\nabla b) = a\nabla b$ and $b\Delta(a\nabla b) = b$. ∎

Corollary 3. If (Q,∇,Δ) is a near-lattice and $b \leq_\nabla a$, then $b \leq_\nabla a\Delta b \leq_\nabla a$. ∎

Corollary 3'. If (Q,∇,Δ) is a near-lattice and $b \leq_\Delta a$, then $b \leq_\Delta a\nabla b \leq_\Delta a$. ∎

Corollary 4. If (Q,∇,Δ) is a near-lattice, then

(a) $a \leq_\nabla a\Delta(a\nabla b) \leq_\nabla a\nabla b$,

(b) $b \leq_\nabla b\Delta(a\nabla b) \leq_\nabla a\nabla b$,

(c) $a\Delta b \leq_\Delta a\nabla(a\Delta b) \leq_\Delta a$,

(d) $a\Delta b \leq_\Delta b\nabla(a\Delta b) \leq_\Delta b$. ∎

Proposition 5. Let (Q,∇,Δ) be a near-lattice. If for $a,b\in Q$

(*) $a\Delta(b\nabla a) = b\Delta(a\nabla b)$, then

(**) $a\Delta b = a\nabla b = a\Delta(b\nabla a) = b\Delta(a\nabla b) = (a\Delta b)\nabla a = (b\Delta a)\nabla b$.

Proof. Let a and b be comparable in any of two semilattices, for instance let $a\nabla b = b$. Then, using (*), $a\Delta b = b$, and hence we have (**).

Let a and b be incomparable. By Corollaries 3 and 3', since $a\Delta b <_{\Delta} a$ and $a\Delta b <_{\Delta} b$,

$$a\Delta b \le_{\Delta} (a\Delta b)\nabla b \le_{\Delta} b.$$

Since $(a\Delta b)\nabla a = (a\Delta b)\nabla b$, we have

$(a\Delta b)\nabla a = (a\Delta b)\nabla b = a\Delta b$. Dually,

$(a\nabla b)\Delta a = (a\nabla b)\Delta b = a\nabla b$, hence, we have (**). ■

Lemma 6. If (Q,∇,Δ) is a near-lattice, then

(a) For all $a,b\in Q$, if $a\Delta b \le_{\nabla} a$ and $b\nabla(a\Delta b) = a\nabla b$, then $a\Delta b = a$.

(b) For all $a,b\in Q$, if $(a\Delta b)\nabla b = a\nabla b$ and $a \le_{\nabla} (a\Delta b)\nabla a \le_{\nabla} a\nabla b$, then $(a\Delta b)\nabla a = a\Delta b$.

(c) For all $a,b\in Q$ if $a\nabla b = (a\Delta b)\nabla a = (a\Delta b)\nabla b$, then $a\nabla b = a\Delta b$.

Proof.

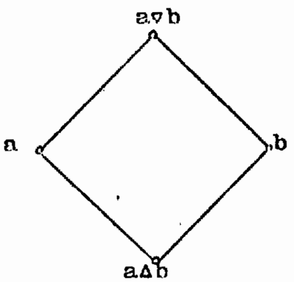
(a) If $a\Delta b \le_{\Delta} a$, then $a = (a\Delta b)\nabla a = a\Delta(b\nabla a)$. Since $a\nabla b = b\nabla(a\Delta b) = (b\nabla a)\Delta b$, we have that

$$a\Delta b = (a\Delta(b\nabla a))\Delta b = a\Delta((b\nabla a)\Delta b) = a\Delta(a\nabla b) = a.$$

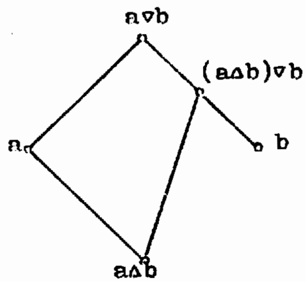
(b) From $a\nabla b = (a\Delta b)\nabla b = b\Delta(a\nabla b)$, it follows that $a\nabla b \le_{\Delta} b$. By Corollary 4(c), since $a\Delta b \le_{\Delta} a\Delta(a\nabla b) \le_{\Delta} a$, it follows that $a\Delta b \le_{\Delta} a\Delta(a\nabla b) \le_{\Delta} a\Delta b$, hence $(a\Delta b)\nabla a = a\Delta(a\nabla b) = a\Delta b$.

(c) From $a\nabla b = (a\Delta b)\nabla a = a\Delta(b\nabla a)$, and $a\nabla b = (a\Delta b)\nabla b = b\Delta(a\nabla b)$, it follows that $a\nabla b \le_{\Delta} a$, and $a\nabla b \le_{\Delta} b$, hence $a\nabla b \le_{\Delta} a\Delta b$. By Corollary 4(c) $a\Delta b \le_{\Delta} a\nabla b$, hence $a\nabla b = a\Delta b$. ■

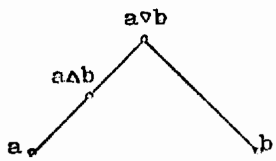
Lemma 7. Let (Q,∇,Δ) be a near-lattice, and $a,b\in Q$. If a and b are incomparable under $\le_{\nabla}$, then there are 6 up to the isomorphism different subsemilattices generated by a, b and $a\Delta b$ in the semilattice (Q,∇). (See the diagrams in Figures 2.1-2.6).



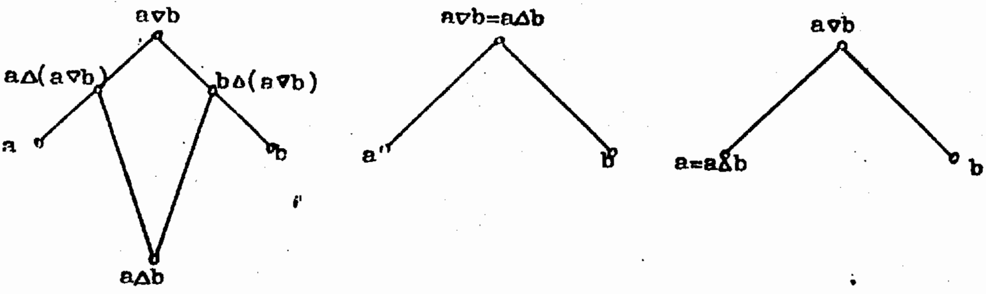
2.1



2.2



2.3



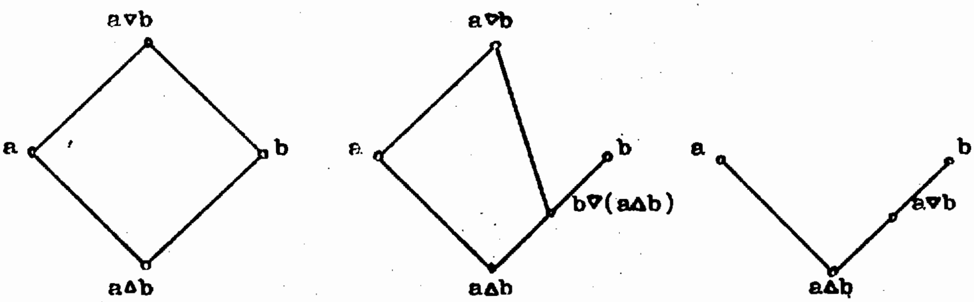
2.4 2.5 2.6

Fig.2

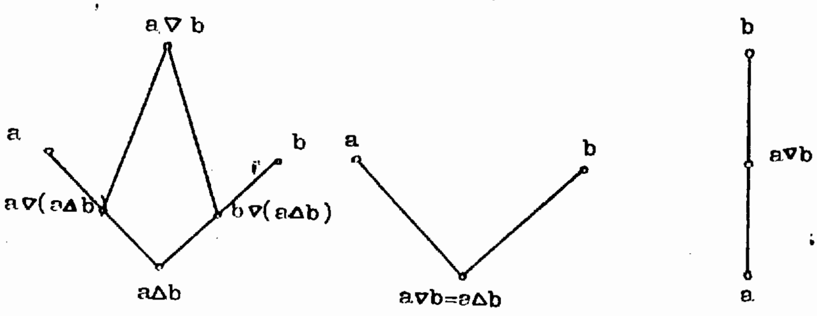
Proof.

By Corollaries 4(a) and 4(b), $a \leq_\nabla (a\Delta b)\nabla a \leq_\nabla a\nabla b$, and $b \leq_\nabla b\nabla(a\Delta b) \leq_\nabla a\nabla b$. There are nine cases: $a\nabla(a\Delta b) = a$, $a <_\nabla a\nabla(a\Delta b) <_\nabla a\nabla b$ and $a\nabla(a\Delta b) = a\nabla b$, combined with cases $b\nabla(a\Delta b) = b$, $b <_\nabla b\nabla(a\Delta b) <_\nabla a\nabla b$ and $b\nabla(a\Delta b) = a\nabla b$. Also, we differ cases when $a\nabla(a\Delta b) = a\Delta b$, and $a\nabla(a\Delta b) \neq a\Delta b$. Using Lemma 6 we have that subsemilattices represented in Figures 2.1-2.6 are the only, up to the isomorphism, subsemilattices generated by a,b and $a\Delta b$ in the semilattice (Q,∇). ■

Lemma 8. Let (Q,∇,Δ) be a near-lattice and $a,b\in Q$. If a,b and $a\Delta b$ generate subsemilattices of the semilattice (Q,∇) represented in Figures 2.1-2.6, respectively, then a,b and $a\nabla b$ generate subsemilattices of the semilattice (Q,Δ) represented in Figures 3.1-3.6, respectively.



3.1 3.2 3.3



3.4 3.5 3.6

Fig.3

Proof.

-The case in Fig.2.1

From $a = a\nabla(a\Delta b) = a\Delta(a\nabla b)$ and $b = b\nabla(a\Delta b) = b\Delta(a\nabla b)$, it follows that $a \leq_\Delta a\nabla b$ and $b \leq_\Delta a\nabla b$. Since a and b are incomparable under $\leq_\Delta$ (otherwise we would have $a\Delta b = a$ or $a\Delta b = b$, which is not case), the only suitable subsemilattice is the one in Fig.3.1.

-The case in Fig.2.2

$a = (a\Delta b)\nabla a = a\Delta(b\nabla a)$, and $(a\Delta b)\nabla b = (a\nabla b)\Delta b$, and the element $(a\Delta b)\nabla b$ differs from $a\Delta b$ and from b, hence the only suitable subsemilattice is the one in Fig.3.2.

-The case in Fig.2.3

From $a\Delta b = a\nabla(a\Delta b) = a\Delta(a\nabla b) = a\Delta b\Delta(a\nabla b)$ it follows that $a\Delta b <_\Delta a\nabla b$, and from $a\nabla b = (a\Delta b)\nabla b = b\Delta(a\nabla b)$, we have that $a\nabla b <_\Delta b$, hence the only suitable subsemilattice is the one in Fig. 3.3.

-The case in Fig.2.4

Elements $(a\Delta b)\nabla a = a\Delta(b\nabla a)$ and $(b\Delta a)\nabla b = b\Delta(a\nabla b)$ differ from a, b, $a\nabla b$ and $a\Delta b$, hence the only suitable subsemilattice is the one in Fig.3.4.

-The case in Fig.2.5 follows from $a\Delta b=a\nabla b$, and the case in Fig.2.6 follows from Lemma 2. ■

Theorem 9. Necessary and sufficient condition under which a bisemilattice (Q,∇,Δ) is a near-lattice is that each pair a,b of elements of Q determines two related subsemilattices in the semilattices (Q,∇) and (Q,Δ), the pairs of related subsemilattices being represented in figures 4.1-4.9.

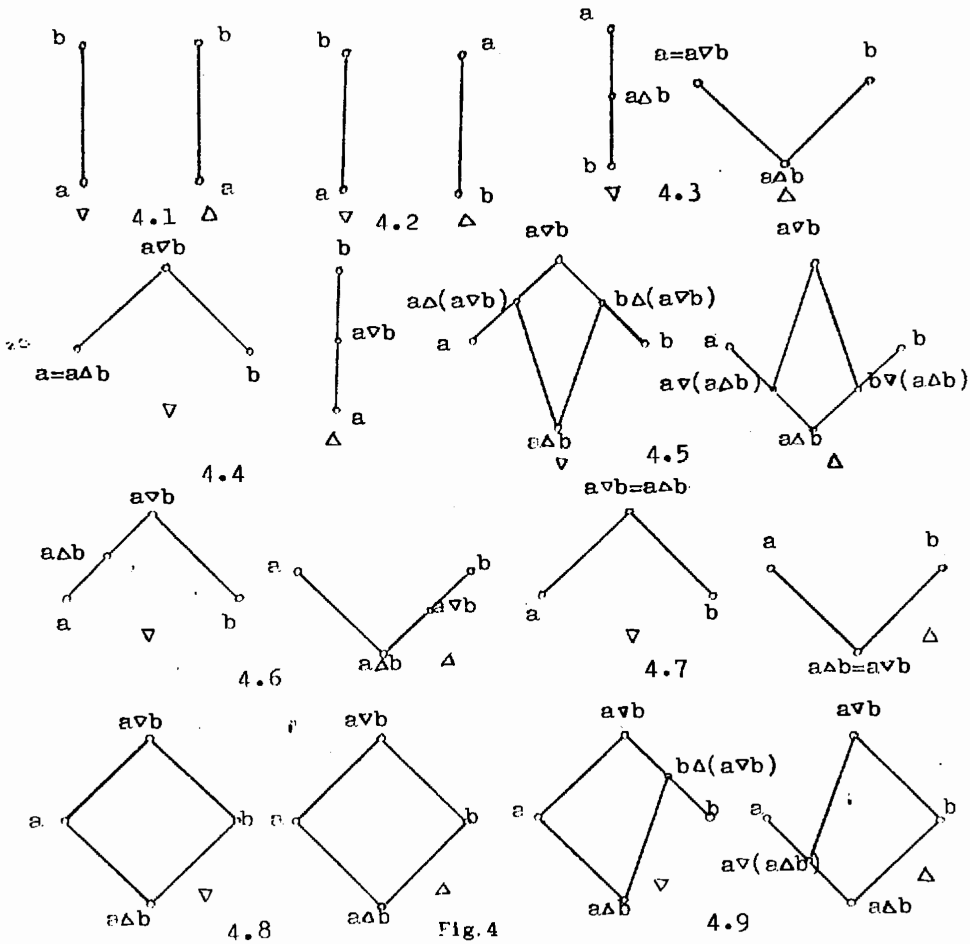


Fig. 4

Proof. By Lemmas 2,6,7 and 8 we have that if (Q,∇,Δ) is a near-lattice, then all $a,b\in Q$ generate a pair of related subsemilattices from Figures 4.1-4.9.

Conversely, since identity (SM) has only two variables and every pair of elements generates two related subsemilattices, we have that every pair satisfies (SM), hence (Q,∇,Δ) is a near-lattice. ■

Corollary 10. If semilattices (Q,∇) and (Q,Δ) are chains, then (Q,∇,Δ) is a near-lattice.

In Examples 4 and 5 we give all, up to the isomorphism, different near-lattices with 3 and 4 elements.

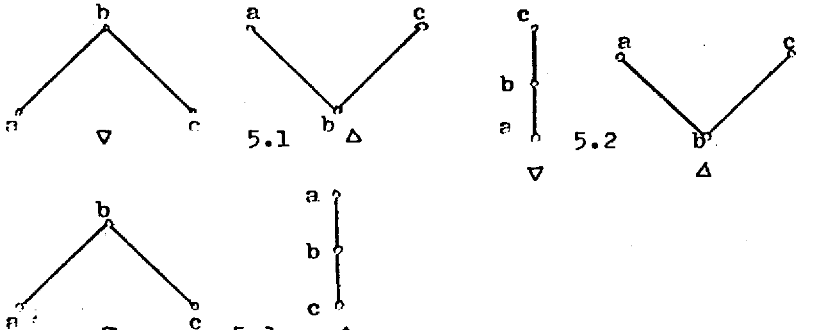
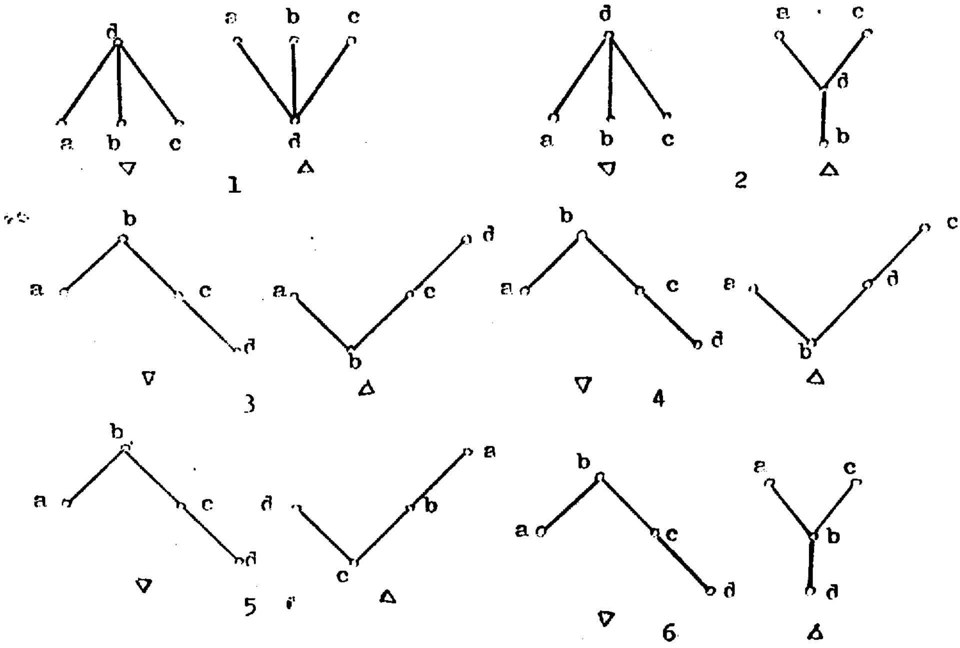
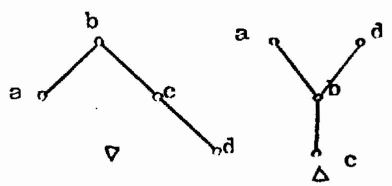


Fig.5

Including near-lattices consisting of two chains, there are 9 nonisomorphic near-lattices with 3 elements.

Example 5. All nonisomorphic near-lattices with four elements, except these consisting of two chains, are presented in Fig.6.





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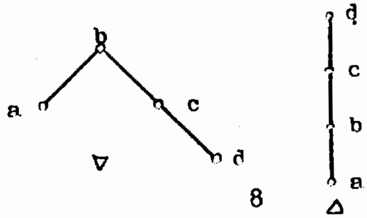

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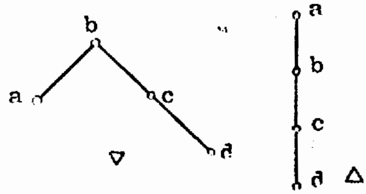

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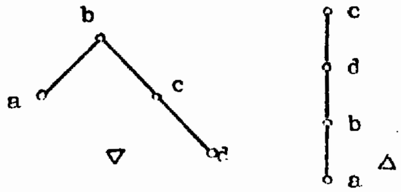

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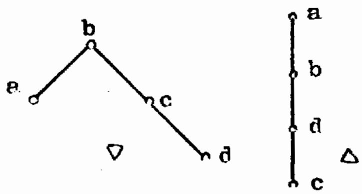

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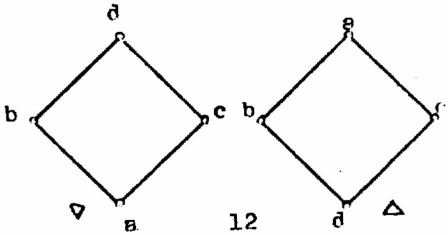

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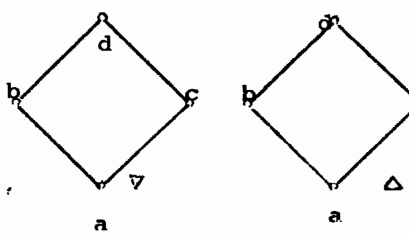

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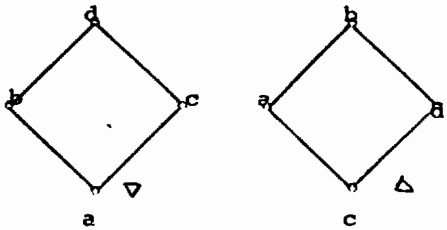

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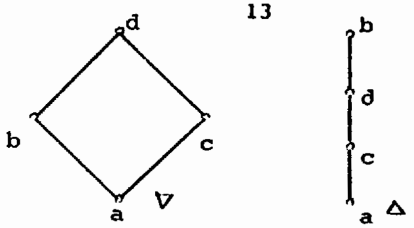

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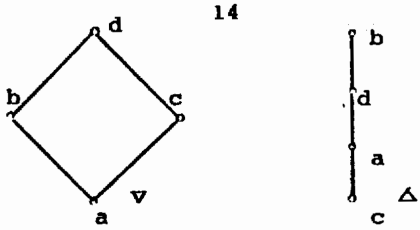

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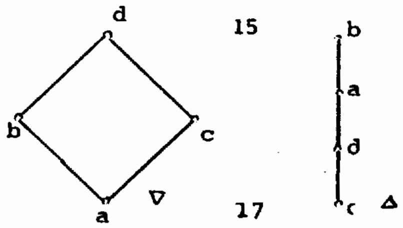

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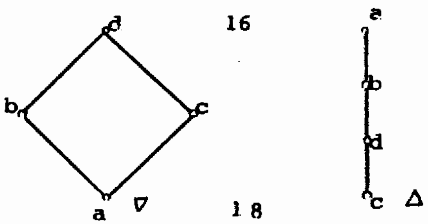

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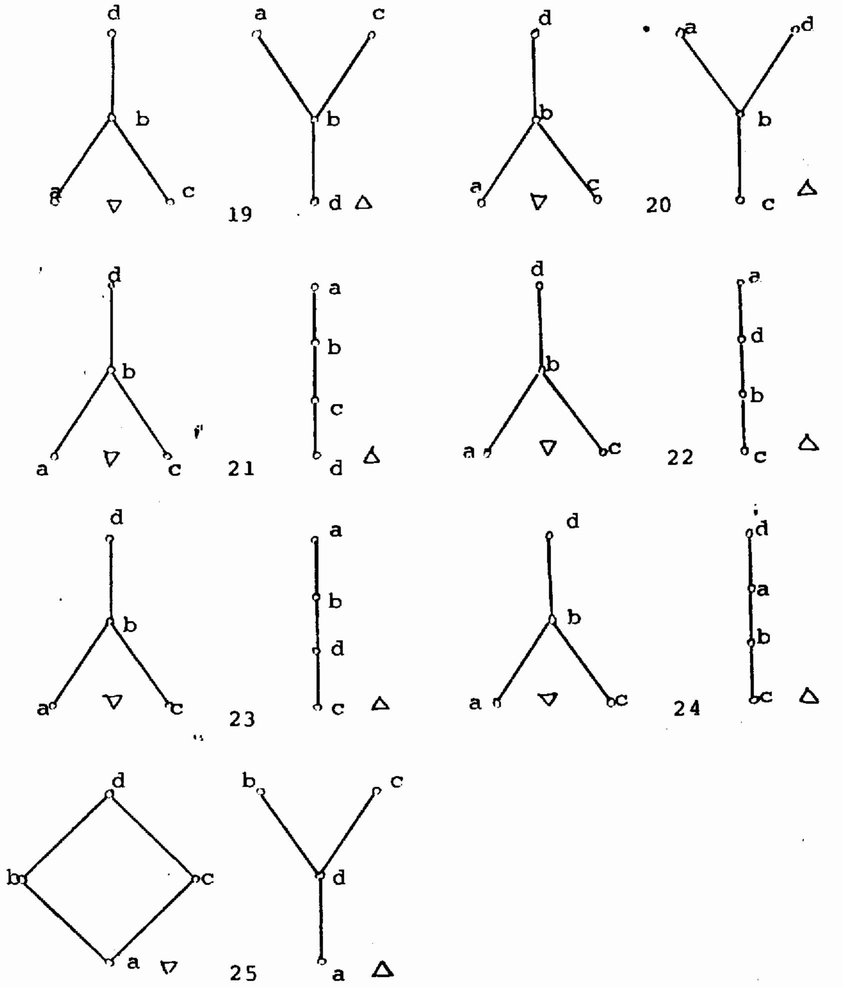

Fig.6

Including near-lattices dual to the ones denoted by 2,6,7,8,9,10,11,15,16,17,18,21,22,23,24 and 25, and those consisting of two chains, there are 65 nonisomorphic near-lattices with 4 elements.

Near-lattices and some other classes of bisemilattices are related as in Fig.7.

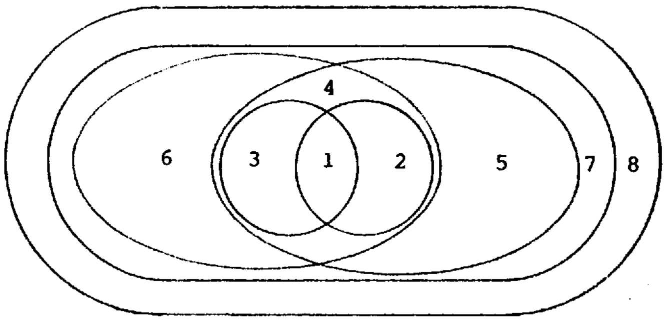

Fig.7

1. distributive lattices
2. lattices
3. distributive bisemilattices
4. bisemilattices which are both, (∇,Δ)-weak-distributive, and (Δ,∇)-weak-distributive
5. (∇,Δ)-weak-distributive bisemilattices
6. (Δ,∇)-weak-distributive bisemilattices
7. near-lattices
8. bisemilattices.

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O JEDNOJ KLASI BIPOLUMREŽA
R E Z I M E

U radovima [1] i [2] je uveden pojam skoro-mreže, kao bipolumreže (Q,∇,Δ) u kojoj važi zakon $x\Delta(y\nabla z\nabla x)=(x\Delta y)\nabla(x\Delta z)\nabla(x\Delta x)$. Na ovom mestu takva struktura nazvana je (Δ,∇)-slabo-distributivna bipolumreža, a struktura u kojoj važi dualni zakon (∇,Δ)-slabo-distributivna bipolumreža. U ovom radu definisan je pojam skoro-mreže kao bipolumreže u kojoj važi zakon $x\nabla(y\Delta x) = (x\nabla y)\Delta x$, dokazana su neka svojstva takvih skoro-mreža i utvrđen jedan potreban i dovoljan uslov da bipolumreža bude skoro-mreža.

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