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CEP AND HOMOMORPHIC IMAGES OF ALGEBRAS

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Abstract

The CEP (congruence extension property) of an algebra is characterized by means of modular pairs in its lattice of weak congruences. Hence the necessary and sufficent conditions are deduced under which all homomorphic images of an algebra having the CEP, satisfy the same property. This characterization is purely lattice theoretic, so two special cases are considered for which some sufficent conditions (modularity of congruence lattice, for example) are given.

Introduction

If $\mathcal{A} = (A,F)$ is an algebra, then $C_w\mathcal{A}$ is the lattice of all the weak congruences on $\mathcal{A}$, i.e. of all the symmetric and transitive subalgebras of $\mathcal{A}^2$. $C_w\mathcal{A}$ coincides with the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ (under set inclusion). Moreover, Con$\mathcal{A}$ is its sublattice (as a filter generated by $\Delta = \{(x,x) \mid x \in B\}$), and Sub$\mathcal{A}$ is a retract of $C_w\mathcal{A}$ (as an ideal generated by Δ). In particular, subalgebras are represented by diagonal relations in $C_w\mathcal{A}$ ($B \in$ Sub$\mathcal{A}$ corresponds to $d_{B^2} = \{(x,x) \mid x \in B\}$). (For the notation, definitions, and other properties of $C_w\mathcal{A}$, see [1], [2], [3], and the references given there).

If L is any lattice, and $a,b \in L$, then (a,b) is said to be the *modular pair* in L (see for example [4]), if for every $x \in L$,

$$x \le b \text{ implies } x \vee (a \wedge b) = (x \vee a) \wedge b.$$

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A modular pair (a,b) is often denoted by aMb. Obviously, L is modular if and only if aMb for all $a,b \in L$.

Theorem 1. *An algebra* $\mathcal{A}$ *has the CEP if and only if for every* $B \in \mathrm{Sub}\mathcal{A}$, $\Delta M B^2$ *in* $C_w\mathcal{A}$ (that is, iff for every $\rho \in C_w\mathcal{A}$

$$\rho \le B^2 \text{ implies } \rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2,$$

where $\Delta = \{(x,x) \mid x \in \mathcal{A}\}$).

Proof.

It is well known that $\mathcal{A}$ has the CEP if every congruence ρ on a subalgebra B of $\mathcal{A}$ can be extended to the congruence Θ on $\mathcal{A}$, i.e. iff

$$\rho = \Theta \cap B^2 \text{ , which is equivalent to}$$

$$\rho = (\rho \vee \Delta) \wedge B^2 \quad \text{in } C_w\mathcal{A}\ . \tag{1}$$

Indeed, if there is such a Θ, then there is at least one with the same property, namely $\Theta = \rho \vee \Delta$; on the other hand, if (1) is satisfied, then $\Theta = \rho \vee \Delta$.

Now, (1) is equivalent to

$$\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2, \tag{2}$$

since $\rho \ge \Delta \wedge B^2$, and hence $\rho = \rho \vee (\Delta \wedge B^2)$.

Thus, for $\rho \in \mathrm{Con}B$, the CEP is equivalent to

$$\rho \le B^2 \longrightarrow \rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2\ . \tag{3}$$

Even in the case when $\rho \in \mathrm{Con}C$, $C < B$ (we still have $\rho \le B^2$), the CEP implies (3). Indeed, if $\sigma = \rho \vee (\Delta \wedge B^2)$ (which belongs to $\mathrm{Con}B$), then by the CEP

$$\sigma = (\sigma \vee \Delta) \wedge B^2, \text{ i.e.}$$

$$\rho \vee (\Delta \wedge B^2) = (\rho \vee (\Delta \wedge B^2) \vee \Delta) \wedge B^2, \text{ and}$$

since $\rho \vee \Delta = (\rho \vee (\Delta \wedge B^2)) \vee \Delta$ (see [1]), finally we have

$$\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2\ .$$

Thus, the CEP implies (3) for every $\rho \in C_w\mathcal{A}$. Since the converse has already been proved, we are done. □

It is well known that for $\Theta \in \mathrm{Con}\mathcal{A}$, the lattice $\mathrm{Con}\mathcal{A}/\Theta$ is isomorphis with the filter $[\Theta)$ in $\mathrm{Con}\mathcal{A}$, under $\rho \to \rho/\Theta$, where $[a]_\Theta(\rho/\Theta)[b]_\Theta$ iff $(a,b) \in \rho$. Hence, the lattice $C_w\mathcal{A}/\Theta$ ($\Theta \in \mathrm{Con}\mathcal{A}$) consists (up to the above isomorphism) of all the intervals $[\Theta \wedge B^2, B^2]$ in $C_w\mathcal{A}$, with the property $B[\Theta]$

isomorphism) of all the intervals $[\Theta \wedge B^2, B^2]$ in $C_w\mathcal{A}$, with the property $B[\Theta] = B$, where $B[\Theta] = \{x \in A \mid x\Theta b \text{ for some } b \in B\}$.

In the following we shall identify those intervals and their inverse images.

Theorem 2. *Let $\Theta \in Con\mathcal{A}$. Then $\mathcal{A}/\Theta$ has the CEP if and only if in the lattice $C_w\mathcal{A}\Theta MB^2$, for every $B \in Sub\mathcal{A}$ satisfying $B = B[\Theta]$*. (In other word, $\mathcal{A}/\Theta$ has the CEP iff for every such $\mathcal{B}, \rho \leq B^2$ implies $\rho \vee (\Theta \wedge B^2) = = (\rho \vee \Theta) \wedge B^2$).

Proof. By the above mentioned isomorphism, $\mathcal{A}/\Theta$ has the CEP if and only if for every $\mathcal{B} \in Sub\mathcal{A}$, such that $B = B[\Theta]$, and for every $\rho \in [B^2 \wedge \Theta, B^2]$, there is $\sigma \in [\Theta)$ such that $\rho = \sigma \cap B^2$, i.e. if and only if

(1) $$\rho = (\rho \vee \Theta) \wedge B^2 \text{ in } C_w\mathcal{A} .$$

Obviously, $\rho = \rho \vee (\Theta \wedge B^2)$, and thus (1) is equivalent with

(2) $$\rho \vee (\Theta \wedge B^2) = (\rho \vee \Theta) \wedge B^2,$$

where

$$B^2 \wedge \Theta \leq \rho \leq B^2 .$$

If ρ is any (weak) congruence satisfying $\rho \leq B^2$, then, obviously

(3) $$\rho \vee \Theta = (\rho \vee (\Theta \wedge B^2)) \vee \Theta .$$

Now, if $\mathcal{A}/\Theta$ has the CEP, then for $\sigma = \rho \vee (\Theta \wedge B^2)$, we have

$$\sigma = (\sigma \vee \Theta) \wedge B^2, \quad \text{i.e.}$$

$$\rho \vee (\Theta \wedge B^2) = (\rho \vee (\Theta \wedge B^2) \vee \Theta) \wedge B^2 .$$

Hnece, by (3)

$$\rho \vee (\Theta \wedge B^2) = (\rho \vee \Theta) \wedge B^2 , \text{ for } \rho \leq B^2 ,$$

which was to be proved. □

Since every homomorphic image of $\mathcal{A}$ is isomorphic with $\mathcal{A}/\Theta$ for some $\Theta \in Con\mathcal{A}$, Theorem 2 gives a characterization of the CEP for any homomorphic image of $\mathcal{A}$. To extend this characterization on every interval $[\Theta \wedge B^2, B^2]$ in $C_w\mathcal{A}$ (not only on those satisfying condition $B = B[\Theta]$), we need the following lemma.

Lemma 3. *Let $\mathcal{A}$ have the CEP, and let $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$,$\mathcal{B} < \mathcal{C}$. If $\rho \in Con\mathcal{B}$, $\Theta \in Con\mathcal{A}$, and if $\rho < B^2 \wedge (\rho \vee \Theta)$, then $\rho \vee (\Delta \wedge C^2) < C^2 \wedge (\rho \vee \Theta)$, all in the lattice $C_w\mathcal{A}$.*

Proof. Suppose that

(1) $$\rho \vee (\Delta \wedge C^2) = C^2 \wedge (\rho \vee \Theta) .$$

By the CEP, and since $\mathcal{B} < C$, we have

(2) $$\rho = B^2 \wedge (\rho \vee (\Delta \wedge C^2)) .$$

From (2) and (1), we finally get

$$\rho = B^2 \wedge C^2 \wedge (\rho \vee \Theta) = B^2 \wedge (\rho \vee \Theta) .$$

The proof now follows by contraposition. □

Theorem 4. *Let $\mathcal{A}$ be an algebra satisfying $B[\Theta] < A$ for every $\Theta \in Con\mathcal{A}$, $\mathcal{B} \in Sub\mathcal{A}$, $\Theta \neq A^2$, $B \neq A$. Then, every homomorphic image of $\mathcal{A}$ has the CEP, if and only if $\Theta\ M\ B^2$ for every $\Theta \in Con\mathcal{A}$, and for every $\mathcal{B} \in Sub\mathcal{A}$ (all in $C_w\mathcal{A}$).*

Proof. By the proof of Theorem 2, the CEP on $\mathcal{A}/\Theta$ is equivalent with $\Theta M B^2$ for every $\mathcal{B}$ such that $B = B[\Theta]$. Now, if $\Theta M B^2$ is not satisfied for some subalgebra $\mathcal{B}$, such that $B < B[\Theta]$, then it is not true that $\Theta M C^2$ for any $C \in$ Sub$\mathcal{A}$, such that $\mathcal{B} \leq C$ (by assumption on $\mathcal{A}$, such a $C \neq \mathcal{A}$ always exists). This contradicts Theorem 2, proving that $\Theta\ M\ B^2$ for every $\mathcal{B} \in$ Sub$\mathcal{A}$. □

It is possible to connect the given lattice characterizations of the CEP on homomorphic images of $\mathcal{A}$, with some known properties of Con$\mathcal{A}$. But first, we need the following definition.

An algebra $\mathcal{A}$ has the *weak congruence intersection property* (WCIP), if for $\rho \in C_w\mathcal{A}$ the following implication holds in $C_w\mathcal{A}$:

$$\Delta \leq \Theta \longrightarrow \Delta \vee (\rho \wedge \Theta) = (\Delta \vee \rho) \wedge \Theta .$$

Obviously, $\Delta \leq \Theta$ means that $\Theta \in$ Con$\mathcal{A}$.

Remark. The WCIP is a weakened *congruence intersection property* (CIP); $\mathcal{A}$ has the CIP if for all $\rho,\Theta \in C_w\mathcal{A}$, the following equality holds in $C_w\mathcal{A}$:

$$(\rho \wedge \Theta) \vee \Delta = (\rho \vee \Delta) \wedge (\Theta \vee \Delta) .$$

The CIP was defined in [1] as one of the necessary and sufficent conditions for the modularity of $C_w\mathcal{A}$.

It is clear that the CIP implies the WCIP, since for $\Theta \in \mathrm{Con}\mathcal{A}$, $\Theta \vee \Delta = \Theta$.

Theorem 5. *If $\mathcal{A}$ has the WCIP and $\mathrm{Con}\mathcal{A}$ is modular, then the CEP is hereditary for homomorphic images of $\mathcal{A}$.*

Proof. Let $\rho \leq B^2$, $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, $\Theta \in \mathrm{Con}\mathcal{A}$. Then,

$(\rho \vee (\Theta \wedge B^2)) \vee \Delta = (\rho \vee \Delta) \vee ((\Theta \wedge B^2) \vee \Delta) =$ (by WCIP) $=$

$(\rho \vee \Delta) \vee (\Theta \wedge (B^2 \vee \Delta)) =$ (since $\mathrm{Con}\mathcal{A}$ is modular) $=$

$((\rho \vee \Delta) \vee \Theta) \wedge (B^2 \vee \Delta) = (\rho \vee (\Delta \vee \Theta)) \wedge (B^2 \vee \Delta) =$

$(\rho \vee \Theta) \wedge (B^2 \vee \Delta) =$ (again by WCIP) $= ((\rho \vee \Theta) \wedge B^2) \vee \Delta$.

$\mathcal{A}$ has the CEP, and $\rho \vee (\Theta \wedge B^2)$ as well as $(\rho \vee \Theta) \wedge B^2$ belong to $\mathrm{Con}\mathcal{B}$. $\rho \to \rho \vee \Delta$ is then an injection (see [1]), and thus

$$\rho \vee (\Theta \wedge B^2) = (\rho \vee \Theta) \wedge B^2 \ . \ \square$$

The CEP can be hereditary for homomorphic images even if $\mathrm{Con}\mathcal{A}$ is not modular. To prove this, we need the following lemma.

Lemma 6. *If $\mathcal{A}$ has the CEP and WCIP, then for $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$, $\Theta \in \mathrm{Con}\mathcal{A}$, the following implication holds in $C_w\mathcal{A}$:*

$$\Theta \leq B^2 \vee \Delta \longrightarrow \text{for } \rho \in [B^2 \wedge \Theta, B^2],\ \rho \vee \Theta = \rho \vee \Delta \ .$$

Proof. If $\Theta \leq B^2 \vee \Delta$ and $\rho \in [B^2 \wedge \Theta, B^2]$, then

$$\Theta = (\Delta \vee B^2)\ \ \Theta = \text{(by WCIP)} = \Delta \vee (B^2 \wedge \Theta) \leq \rho \vee \Delta \ .$$

Hence $\rho \vee \Delta \vee \Theta = \rho \vee \Delta$, i.e. $\rho \vee \Theta = \rho \vee \Delta$. $\square$

Theorem 7. *Let $\mathcal{A}$ have the WCIP, and for every $\mathcal{B} \in \mathrm{Sub}\mathcal{A}$ $\mathrm{Con}\mathcal{A} \setminus [B^2 \vee \Delta) = (B^2 \vee \Delta]$, then the CEP is hereditary for homomorphic images of $\mathcal{A}$.* ($[B^2 \vee \Delta)$ is a filter and $(B^2 \vee \Delta]$ an ideal generated by $B^2 \vee \Delta$ respectively, both in $\mathrm{Con}\mathcal{A}$.)

Proof. $\mathcal{A}$ has the WCIP, and since nontrivial factor-algebras are generated by congruences less than $B^2 \vee \Delta$ only, the conditions of Lemma 6 are satisfied. Hence, for $\Theta \in \mathrm{Con}\mathcal{A}$, $\Theta \leq B^2 \vee \Delta$, $\rho \in \mathrm{Con}\mathcal{B}$, since $\Theta \wedge B^2 \leq \rho$,

$$(\rho \vee \Theta) \wedge B^2 = (\rho \vee \Delta) \wedge B^2 = \rho = \rho \vee (\Theta \wedge B^2) \ . \ \square$$

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Rezime

CEP I HOMOMORFNE SLIKE ALGEBRI

U radu se daju potrebni i dovoljni uslovi da se svojsvo proširenja kongruencija (CEP) prenese sa algebre na sve njene homomorfne slike. Uslovi su formulisani mrežnim zakonima u mreži slabih kongruencija algebre. Daju se i neke posledice tih uslova.

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