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THE FUZZY POWER OF ALGEBRAS

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Abstract

In the usual constructions of fuzzy algebraic structures, the lattice being the range of all the mappings is complete and often distributive. Complementedness (without distributivity) is rarely used in such constructions.

It is shown that a class of complemented lattices can be used to construct a lattice extension of an algebra and its fuzzy subalgebras.

1. Let L be a complete lattice with a zero (0) and unit (1) element, and $\mathcal{A}=(A,F)$ an arbitrary algebra. Let $A(L)$ be a collection of L-fuzzy sets on A (i.e. of all the mappings $\bar{X}\colon A \to L$), such that

1) $\bar{X}(a) \wedge \bar{X}(b) = 0$, for all $a,b \in A$, $a \neq b$;

2) $\bigvee_{a \in A} \bar{X}(a) = 1$.

Define the operations on $A(L)$ in the same way as for the Boolean power ([1]): if $f \in F_n \subseteq F$, and $\bar{X}_1,\dots,\bar{X}_n \in A(L)$, then

$f(\bar{X}_1,\dots,\bar{X}_n) = \bar{Y}$, and for $a \in A$

$\bar{Y}(a) = \bigvee (\bar{X}_1(a_1) \wedge \dots \wedge \bar{X}_n(a_n)$; $f(a_1,\dots,a_n = a)$.

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We shall say that L *allows the fuzzy power* if for an arbitrary algebra $\mathcal{A}$,

(i) every $p \in L$ belongs to the image of some $\overline{X} \in A(L)$, and

(ii) 3) defines an operation on $A(L)$, i.e.

$$f(\overline{X}_1,\dots,\overline{X}_n) \in A(L), \text{ for all } \overline{X}_1,\dots,\overline{X}_n \in A(L) .$$

The algebra $(A(L),F)$ (briefly $A(L)$) is said to be a *fuzzy power of* $\mathcal{A}$.

2. In the following, each set $P = \{p_i;\ i \in I\}$ L of pairwise-disjoint elements, the supremum of which is 1, will be called *a partition in* L ($p_i \wedge p_j = 0$, $\vee\, p_i = 1$). (A partition is obviously a maximal orthogonal system in L.)

It is clear that for an $\overline{X} \in A(L)$, the set $\{X(a);\ a \in A\}$ (the image of $\overline{X}$) is a partition in L.

Theorem 1. *A complete lattice L allows the fuzzy power if and only if it satisfies the following three conditions:*

a) L is complemented;

b) If $\{p_i;\ i \in I\}$ is a partition in L, then $\{q_j;\ j \in J\}$ is also a partition in L, where J is any set-theoretic partition of I, and for every $j \in J$, $q_j = \bigvee_{k \in j} p_k$;

c) If $P_1,\dots,P_n$ are partitions in L, then

$P = \{p_1 \wedge \dots \wedge p_n;\ p_i \in P_i,\ i = 1,\dots,n\}$ is also a partition in L.

Proof.

($\rightarrow$) Let L be a lattice which allows the fuzzy power. Then:

a) L is complemented. Indeed, by (i) every $p \in L$ belongs to the image of some $\overline{X}$, that is, to one partition, say $\{p_i;\ i \in I\}$. By (ii), for n=1, i.e. for unary operations, $\{p, \bigvee_{p_i \neq p} p_i\}$ is also a partition in L, and $\bigvee_{p_i \neq p} p_i$ is a complement of p.

b) Consider an arbitrary partition P. Since (ii) holds for any unary operation applied on some $\overline{X}$ such that $\{\overline{X}(a);\ a \in A\} = P$, the proof of this part follows immediately.

c) To prove the third property, consider an n-ary operation f, and fuzzy sets $\bar{X}_1,\dots,\bar{X}_n$, such that $\{\bar{X}_i(a);\ a \in A\} = P_i$, $i=1,\dots,n$. The proof now follows from the fact that the image of $f(\bar{X}_1,\dots,\bar{X}_n)$ has to be a partition.

($\leftarrow$) Let L now satisfy a), b) and c). Then, since L is complemented, (i) is satisfied in $A(L)$ for any algebra $\mathcal{A}$. If $f \in F_n$, and $\bar{X}_1,\dots,\bar{X}_n \in A(L)$, then ($ii$) holds, since by b) and by c) $f(\bar{X}_1,\dots,\bar{X}_n)$ is a partition in L. □

Considering lattices with finite partitions only, one can characterize those which allow the fuzzy power by means of 0,1-lattice homomorpshisms, i.e. using the congruences on L with one element minimal and maximal class ($[0]_\theta=\{0\}$, $[1]_\theta=\{1\}$), called in the following 0,1-congruences.

Theorem 2. *If a complete lattice L contains no infinite partition, then L allows the fuzzy power if and only if there is a 0,1-congruence* θ *on a semilattice* $(L,\wedge)$ *such that* $(L/\theta, \wedge, \vee, ', 0, 1)$ *is a Boolean algebra, where*

$$(*)\quad \begin{cases} [x]_\theta \vee [y]_\theta \overset{\text{def}}{=} \begin{cases} [x]_\theta, & \text{if } [x]_\theta = [y]_\theta \\ [x \vee y]_\theta & \textit{otherwise} \end{cases} ; \\ [x]'_\theta = [x']_\theta ; \end{cases}$$

"$\wedge$" *is induced by* "$\wedge$" *in L.*

Proof.

($\leftarrow$) Suppose that there is a congruence θ on $(L,\wedge)$, satisfying (*). Then, $\{p_1,\dots,p_n\}$ is a partition in L if and only if $\{[p_1]_\theta,\dots,[p_n]_\theta\}$ is a partition on L/θ. Indeed, $p \wedge q = 0$ implies $[p]_\theta \neq [q]_\theta$, and thus

$$\bigvee_{i=1}^{n} [p_i]_\theta = \Big[\bigvee_{i=1}^{n} p_i\Big]_\theta = 1, \quad \text{and} \quad [p_i]_\theta \wedge [p_j]_\theta = [p_i \wedge q_j]_\theta = 0 .$$

On the other hand, if $\{[p_1]_\theta,\dots,[p_n]_\theta\}$ is a partition in L/θ, then $[p_i]_\theta \wedge [p_j]_\theta = 0$ implies $[p_i \wedge p_j]_\theta = [0]_\theta$, i.e. $p_i \wedge p_j = 0$, and $1 = \bigvee_{i=1}^{n} [p_i]_\theta = \Big[\bigvee_{i=1}^{n} p_i\Big]_\theta = [1]_\theta = \{1\}$, that is $\bigvee_{i=1}^{n} p_i = 1$, proving that $\{p_1,\dots,p_n\}$ is a partition in L.

Hence, it follows that L allow the power, since the properties a), b) and c) concerning the partitions, are satisfied in the Boolean algebra L/Θ, and thus in L.

($\rightarrow$) Suppose now that L allows the fuzzy power. Define a binary relation Θ on L: $(p,q) \in \Theta$ if and only if there is an $r \in L$, such that $\{p,r\}$ and $\{q,r\}$ are partitions in L. It is clear that Θ is an equivalence relation on L. It is also a congruence on $(L,\wedge)$:
For i=1,2, $(p_i,q_i) \in \Theta$ if and only if there is an $r_i \in L$, such that $\{p_i,r_i\}$ and $\{q_i,r_i\}$ are partitions in L. Then, by c) in Theorem 1,

$\{p_1\wedge p_2, p_1\wedge r_2, p_2\wedge r_1, r_1,r_2\}$ and $\{q_1\wedge q_2, q_1\wedge r_2, q_2\wedge r_1, r_1\wedge r_2\}$

are both partitions in L. Again by c)

$\{p_1\wedge p_2\wedge q_1\wedge q_2, p_1\wedge q_1\wedge r_2, p_2\wedge q_2\wedge r_1, r_1\wedge r_2\}$ is a partition in L.

Moreover, $\{p_1\wedge p_2, (p_1\wedge q_1\wedge r_2) \vee (p_2\wedge q_2\wedge r_1) \vee (r_1\wedge r_2)\}$ and

$\{q_1\wedge q_2, (p_1\wedge q_1\wedge r_2) \vee (p_2\wedge q_2\wedge r_1) \vee (r_1\wedge r_2)\}$ are also two partitions, by b) and by the fact that $p_1\wedge p_2 \geq p_1\wedge p_2\wedge q_1\wedge q_2$, $q_1\wedge q_2 \geq p_1\wedge p_2\wedge q_1\wedge q_2$. Thus, $(p_1\wedge p_2, q_1\wedge q_2) \in \Theta$.

Since $[0]_\Theta = \{p; (0,p) \in \Theta\}$, there is $q \in L$, such that $\{p,q\}$ is a partition, implying that $\{0,q\}$ is also a parition. Hence, q=1, p=0, i.e. $[0]_\Theta=\{0\}$.

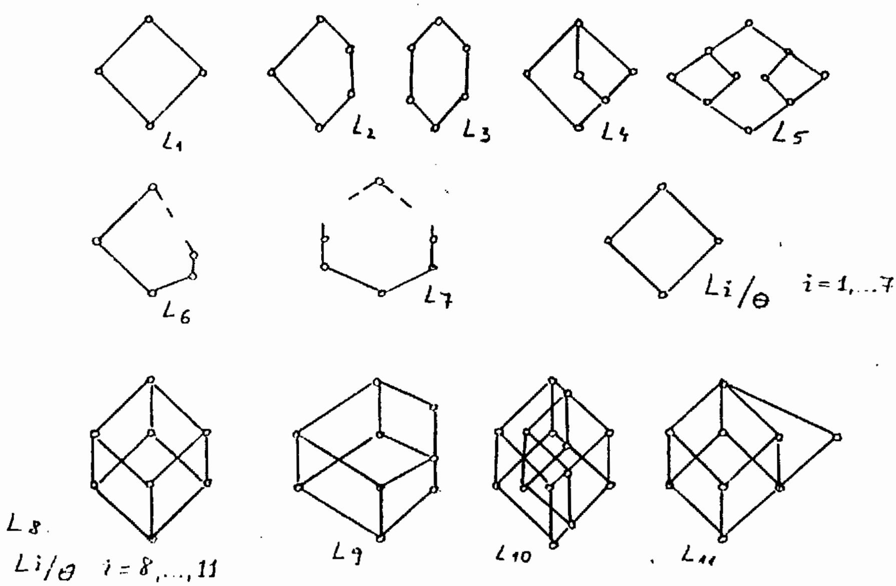
Similarly $[1]_\Theta = \{1\}$.

L/Θ is a a Boolean algebra. Indeed, L/Θ contains the least element 0, and the greatest one 1. It is a uniquely complemented lattice, since for every $p \in L$, there is a complement $p' \in L$ (property a)), and by the definition of Θ, $[p']_\Theta$ contains all the complements of p. To prove that L/Θ is an atomic lattice, take one partition $P = \{p_1,\dots,p_n\}$ in L with a maximal number n of elements (by assumption n is finite). Then, $[P]_\Theta = \{[p_1]_\Theta,\dots,[p_n]_\Theta\}$ is the unique partition in L/Θ with a maximal (n again) number of elements. Indeed, if $Q = \{q_1,\dots,q_n\}$ is another partition in L, then $|P\cdot Q| = |\{p_i\wedge q_j;\ p_i\in P,\ q_j\in Q\}| = n$ (since otherwise $p_{1_1} \geq p_{1_1}\wedge q_{1_1},\dots,p_{1_k} \geq q_{1_k}$ $(k < n)$, and from $\bigvee_{t=1}^{k}(p_{1_t}\wedge q_{1_t}) = 1$, it follows that $\bigvee_{t=1}^{k} p_{1_t} = 1$, and hence $\{p_{1_1}\vee\dots\vee p_{1_k}, P\setminus\{p_{1_1},\dots,p_{1_k}\}\}$ is not a partition in L, contrary to b)). Also, if $|Q| = n$, then $[P]_\Theta = [Q]_\Theta$, since $p_1 \geq p_1\wedge q_1$, $q_1 \geq p_1\wedge q_1,\dots,p_n \geq p_n\wedge q_n$, $q_n \geq p_n\wedge q_n$ and $[p_i]_\Theta = [q_i]_\Theta = [p_i\wedge q_i]_\Theta$, $i=1,\dots,n$.

Now, if r is an arbitrary element from L, then $\{r,r'\}= R$ is a partition in L, and $[R\cdot P]_\Theta= [P]_\Theta$. Hence, either $r \wedge p_1 = 0$, or $[r \wedge p_1]_\Theta= [p_1]_\Theta$, i.e. $[r]_\Theta \wedge [p_1]_\Theta= [p_1]_\Theta$, proving that $[P]_\Theta$ is a set of atoms, and that L/Θ is atomic.

Since, L/Θ is atomic (with n atoms), and uniquely complemented, then it is distributive, and thus it is a (finite) Boolean algebra. □

The Hasse-diagrams of some finite lattices allowing the fuzzy power are given below.



3. The fuzzy power of an algebra $\mathcal{A}$ is defined on the collection of all the fuzzy sets on A. As it is known, we can consider some of these fuzzy sets as fuzzy subalgebras of $\mathcal{A}$ ([2]):

If $\mathcal{A}=(A,F)$ is an algebra and L a complete lattice, then a fuzzy set $\bar{A}$: $A \to L$ is said to be a *fuzzy sublagebra of* $\mathcal{A}$ if for every operation $f \in F_n \subseteq F$ and for $x_1,\dots,x_n \in A$

$$\bar{A}(f(x_1,\dots,x_n)) \geq \bigwedge_{i=1}^{n} \bar{A}(x_i) \ .$$

Is it possible to restrict a fuzzy power of an aglgebra to any of its fuzzy subalgebras? The answer is positive, as shown by the following theorem.

Theorem 3. *Let $\mathcal{A}$ be an algebra, and L a lattice allowing the fuzzy power. Let $\bar{A}$: $A \to L$ be a fuzzy subalgebra of $\mathcal{A}$, and*

$$\bar{A}(L) \overset{\text{def}}{=} \left\{ \bar{X} \in A(L);\ \bar{X}(a) \le \bar{A}(a),\ \text{for every } a \in A \right\}$$

Then, $(\bar{A}(L), F)$ (briefly $\bar{A}(L)$) *is a subalgebra of the fuzzy power $A(L)$.*

(We shall call $\bar{A}(L)$ a *fuzzy power of a fuzzy (sub)algebra* $\bar{\mathcal{A}}$.)

Proof.

Let $\bar{X}_1,\dots,\bar{X}_n \in \bar{A}(L)$ and $f \in F_n$. We have to prove that $f(\bar{X}_1,\dots,\bar{X}_n) = \bar{Y} \in \bar{A}(L)$, i.e. that for every $a \in A$, $\bar{Y}(a) \le \bar{A}(a)$. Indeed, $\bar{Y}(a) = \bigvee(\bar{X}_1(a_1)\wedge\dots\wedge\bar{X}_n(a_n));\ f(a_1,\dots,a_n) = a)$, and for $i=1,\dots,n$, $\bar{X}_i(a_i) \le \bar{A}(a_i)$. Hence, since $\bar{A}$ is a fuzzy subalgebra of $\mathcal{A}$,

$$\bar{Y}(a) = \bigvee(\bar{A}(a_1) \wedge \dots \wedge \bar{A}(a_n);\ f(a_1,\dots a_n) = a) \le$$

$$\le \bar{A}(f(a_1,\dots,a_n)) = \bar{A}(a). \ \square$$

4. Lattices characterized in 2 are supposed to allow the fuzzy power of any algebra. If we restrict our attention to some special classes of algebras, then we can omit some of the required conditions. In particular, to allow the power of finite algebras, L obviously does not have to be complete. Also, the lattice allows the power of unary algebras if and only if it satisfies *a)* and *b)* in Theorem 1. The condition *c)* can be omitted, since by 3) in 1., such a case does not appear if all the operations on $\mathcal{A}$ are unary.

In the following, we shall give some general properties of fuzzy powers.

Theorem 4. *a) Let $f,g \in F_n$, and $h \in F_1$. Then, the following identities are preserved under the construction of fuzzy powers:*

a) $f(x_1,\dots,x_n) = g(x_1,\dots,x_n)$;

b) $g(x_1,\dots,x_n) = h(f(x_1,\dots,x_n))$.

Proof.

a) Let $a \in A$, and $\bar{X}_1,\dots,\bar{X}_n \in A(L)$. Then,

$f(\bar{X}_1,\dots,\bar{X}_n)(a) = \bigvee(\bar{X}_1(a_1)\wedge\dots\wedge\bar{X}_n(a_n)\ ;\ f(a_1,\dots,a_n = a) =$

$\bigvee(\bar{X}_1(a_1)\wedge\dots\wedge\bar{X}_n(a_n)\ ;\ g(a_1,\dots,a_n = a) = g(\bar{X}_1,\dots,\bar{X}_n)(a).$

Thus, $f(\overline{X}_1,\dots,\overline{X}_n) = g(\overline{X}_1,\dots,\overline{X}_n)$.

b) Again, let $a \in A$, $\overline{X}_1,\dots,\overline{X}_n \in A(L)$. Then,

$$h(f(\overline{X}_1,\dots,\overline{X}_n))(a) = \vee\,(f(\overline{X}_1,\dots,\overline{X}_n)(b)\ ;\ h(b) = a) =$$

$$\vee(\vee(\overline{X}_1(b_1)\wedge\dots\wedge\overline{X}_n(b_n)\ ;\ f(b_1,\dots,b_n = b)\ ;\ h(b) = a) =$$

$$\vee(\overline{X}_1(b_1)\wedge\dots\wedge\overline{X}_n(b_n)\ ;\ h(f(b_1,\dots,b_n)) = a) =$$

$$\vee(\overline{X}_1(b_1)\wedge\dots\wedge\overline{X}_n(b_n)\ ;\ g(b_1,\dots,b_n) = a) = g(\overline{X}_1,\dots,\overline{X}_n)(a).$$

Hence $g(\overline{X}_1,\dots,\overline{X}_n) = h(f(\overline{X}_1,\dots,\overline{X}_n))$. □

Remark. The previous proposition shows that the fuzzy power gives the most if applied on unary algebras, since by b) any identity of the form

$$f_{i_1}\dots f_{i_m}(x) = f_{j_1}\dots f_{j_n}(x)\ ,$$

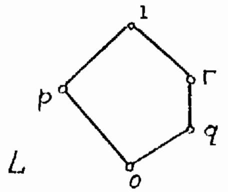
where all the operations are unary, is preserved under the construction of fuzzy powers.

5. We shall conclude with a simple example of a fuzzy power, the algebra being a two-element group. L is here a pentagon η_5.

Example.

A:

·	e	a
e	e	a
a	a	e



$$A(L) = \left(\left\{\begin{bmatrix} e & a \\ 1 & 0\end{bmatrix}, \begin{bmatrix} e & a \\ 0 & 1\end{bmatrix}, \begin{bmatrix} e & a \\ p & q\end{bmatrix}, \begin{bmatrix} e & a \\ q & p\end{bmatrix}, \begin{bmatrix} e & a \\ p & r\end{bmatrix}, \begin{bmatrix} e & a \\ r & p\end{bmatrix}\right\}, \cdot\right)$$

Denote $\begin{bmatrix} e & a \\ p & q\end{bmatrix}$ by pq, etc.

·	10	01	pq	qp	pr	rp
10	10	01	pq	qp	pr	rp
01	01	10	qp	pq	rp	pr
pq	pq	qp	10	01	10	01
qp	qp	pq	01	10	01	10
pr	pr	rp	10	01	10	01
rp	rp	pr	01	10	01	10

(Operation on $A(L)$).

One fuzzy subalgebra $\bar{A}$ of $\mathcal{A}$ is $\bar{A} = \begin{bmatrix} e & a \\ 1 & r \end{bmatrix}$. The corresponding fuzzy power is the following collection of mappings:

$$\bar{A}(L) = \left[\left\{\begin{bmatrix} e & a \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} e & a \\ p & q \end{bmatrix}, \begin{bmatrix} e & a \\ p & r \end{bmatrix}\right\}, \cdot\right]$$

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Rezime

RASPLINUTI STEPEN ALGEBRI

Pokazano je da se jedna klasa komplementiranih mreža može iskoristiti za konstrukciju mrežnih stepena algebri, kao uopštenja Bulovih stepena. Pokazano je da se konstrukcija prirodno proširuje na rasplinute podalgebre i ispitani su neki identiteti koje se pri konstrukciji očuvavaju.

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