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A CLASS OF GENERALIZED RANDOM PROCESSES WITH VALUES IN $L^2(\Omega)$

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Abstract

The structural theorems for generalized random processes from $L(\mathcal{A}_k, Z)$ and Y^{-k} are given.

1. Introduction

In [11] Zemanian introduced the space $\mathcal{A}$, the space of test functions and its dual space $\mathcal{A}'$. Using his ideas we construct a scale of spaces $\mathcal{A}_k$, where k is an integer, whose elements have an orthonormal expansion. Next, we define a generalized random process (g.r.p.) as a continuous linear mapping from $\mathcal{A}_k$ to Z - a separable Hilbert space of random variables with finite second moments. We denote the space of all g.r.p. by $L(\mathcal{A}_k, Z)$. In the definition of g.r.p. we follow [4]. This definition is different from the definitions given in [1,2,7,8,9,10]. In Section 3.2. we construct the space Y^{-k}, where k is an integer, a subspace of $L(\mathcal{A}_k, Z)$. Giving the structural theorems for elements in Y^{-k} and $L(\mathcal{A}_k, Z)$, we establish the relation between them.

Since elements from the spaces $\mathcal{A}_k$ have orthonormal expansions, this enables us to give simpler structural theorems than in [4], where Sobolev

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spaces were observed. The proofs of Theorems 4.1. and 4.2. are given following [4]. The given structural theorems can be applied in solving some classes of stochastic differential equations similar as in [7].

2. Spaces $\mathcal{A}_k$ and $\mathcal{A}'_k$

2.1. We shall follow the notation as in [11, Ch. 9.]. Let I be an open interval in the set of real numbers $\mathbb{R}$, $L^2(I)$ the space of the equivalence classes of square integrable functions with values in the set of complex numbers $\mathbb{C}$. The norm in $L^2(I)$ is defined by

$$\|f\|_0 = \left(\int_I |f(t)|^2 dt\right)^{1/2}$$

Denote by $C^\infty(I)$ the set of infinitely differentiable (smooth) functions and by $\mathbb{N}_0$ and $\mathbb{N}$ sets $\{0,1,2,\dots\}$, $\{1,2,\dots\}$, respectively.

Let $\mathcal{R}$ be a linear differential self-adjoint operator of the form

$$\mathcal{R} = \theta_0 D^{n_1}\theta_1 \dots D^{n_\nu}\theta_\nu ,$$

where $D=d/dx$, n_k, $k=1,2,\dots,\nu$, are non-negative integers θ_k, $k=0,1,\dots,\nu$, smooth complex functions with no zeros on I. Suppose that there exists a sequence of real numbers $\{\lambda_n, n \in \mathbb{N}_0\}$ and a sequence $\{\psi_n, n \in \mathbb{N}_0\}$ of smooth functions in $L^2(I)$ such that $|\lambda_n| \to \infty$ for $n \to \infty$ and

$$\mathcal{R}\psi_n = \lambda_n\psi_n \qquad n \in \mathbb{N}_0 .$$

Furthermore, suppose that $\{\psi_n, n \in \mathbb{N}_0\}$ forms an orthonormal system (o.n.s.) in $L^2(I)$. We can enumerate λ_n and ψ_n so that $|\lambda_0| \le |\lambda_1| \le \lambda_2| \le \dots$
Put

$$\tilde{\lambda}_n = \begin{cases} \lambda_n & \text{if } \lambda_n \neq 0 \\ 1 & \text{if } \lambda_n = 0 \end{cases} \qquad n \in \mathbb{N}_0$$

$\{\tilde{\lambda}_n, n \in \mathbb{N}_0\}$ is a non-decreasing sequence which tends to infinity.

Denote

$$\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k), \quad k \in \mathbb{N}_0 ,$$

where $\mathcal{R}^0 = J$ and J is the identity operator.

Now, we shall define the scale of spaces $\mathcal{A}_k$, $k \in \mathbb{N}_0$. Our construction

$$\mathcal{A}_k = \left\{ \phi \in L^2(I) : \phi = \sum_{n=0}^{\infty} a_n \psi_n ,\ \sum_{n=0}^{\infty} |a_n|^2 \tilde{\lambda}_n^{2k} < \infty \right\}, \quad k \in \mathbb{N}_0 .$$

We see that $\mathcal{A}_0 = L^2(I)$. The space $\mathcal{A}_k$ is the Hilbert space equipped with the scalar product

$$(\phi,\psi)_k = \sum_{n=0}^{\infty} a_n \bar{b}_n \tilde{\lambda}_n^{2k} , \qquad \phi,\psi \in \mathcal{A}_k$$

and the norm

$$\|\phi\|_k = \left(\sum_{n=0}^{\infty} |a_n|^2 \tilde{\lambda}_n^{2k} \right)^{1/2}, \quad \phi \in \mathcal{A}_k ,$$

where $\phi = \sum_{n=0}^{\infty} a_n \psi_n$, $\psi = \sum_{n=0}^{\infty} b_n \psi_n$.

Note that the orthonormal system in $\mathcal{A}_k$ is $\tilde{\psi}_j = \dfrac{\psi_j}{\tilde{\lambda}_j^k}$, $j \in \mathbb{N}_0$.

2. Put

$$S = \left\{ \phi = \sum_{n=0}^{m} a_n \psi_n : m \in \mathbb{N}_0 ,\ \ a_n \in \mathbb{C} \right\} .$$

The set S is dense in $\mathcal{A}_k$, $k \in \mathbb{N}_0$. The operator $\mathcal{R}^m$, $m \in \mathbb{N}_0$ is defined on S. From the fact that the mapping $\mathcal{R}^m$: $S \to L^2(I)$ is linear and continuous, it follow that $\mathcal{R}^m$, $m \le k$ can be extended linearly and continuously to the space $\mathcal{A}_k$. Denote this extension by $\tilde{\mathcal{R}}^m$, $m \le k$. Let $\phi_p = \sum_{n=0}^{\infty} a_n \psi_n \in S$ and $\phi = \sum_{n=0}^{\infty} a_n \psi_n \in \mathcal{A}_k$. We have that $\phi_p \to \phi$, $p \to \infty$, in $\mathcal{A}_k$, so

$$\tilde{\mathcal{R}}^m \phi = \tilde{\mathcal{R}}^m \left[\sum_{n=0}^{\infty} a_n \varphi_n \right] = \lim_{p \to \infty} \left[\mathcal{R}^m \left(\sum_{n=0}^{p} a_n \psi_n \right) \right] = \sum_{n=0}^{\infty} a_n \lambda_n^m \psi_n .$$

Let $\phi \in \mathcal{A}_k \cap C^{\infty}(I)$ and $\langle \mathcal{R}^m \phi,\ \psi_n \rangle = \langle \phi,\ \mathcal{R}^m \psi_n \rangle$, $m \le k$, $n \in \mathbb{N}_0$, where $\langle \phi, \psi \rangle = = \int_I \phi(t)\, \psi(t) dt$, $\phi,\psi \in L^2(I)$. Then $\tilde{\mathcal{R}}^m = \mathcal{R}^m \phi$, $m \le k$.

Next, we shall define tha spaces $\mathcal{A}_{-k}$ $k \in \mathbb{N}_0$ in the following formal way:

$$\mathcal{A}_{-k} = \left\{ f :\ f = \sum_{n=0}^{\infty} b_n \psi_n ,\ \sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} < \infty \right\}, \quad k \in \mathbb{N}_0 .$$

The set $\mathcal{A}_{-k}$ is a vector space (with operations defined in the usual way).

We can define a scalar product and a norm on it.

Namely, let $f = \sum_{n=0}^{\infty} b_n \psi_n$, $g = \sum_{n=0}^{\infty} c_n \psi_n$, then

$$(f,g)_{-k} = \sum_{n=0}^{\infty} b_n \bar{c}_n \tilde{\lambda}_n^{-2k}$$

$$\|f\|_{-k} = \Big(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \Big)^{1/2}$$

It is obvious that $\mathcal{A}_{-k}$ is a Hilbert space.

Let $\mathcal{A}'_k$ be the dual space of $\mathcal{A}_k$, $k \in \mathbb{N}_0$. We have

Theorem 2.1. *There is an isometry between spaces $\mathcal{A}'_k$ and $\mathcal{A}_{-k}$.*

Proof. Let $f \in \mathcal{A}'_k$. Denote by $b_n = (f,\psi_n) = f(\psi_n) = \langle f, \bar{\psi}_n \rangle$, $n \in \mathbb{N}_0$, and let $\phi = \sum_{n=0}^{\infty} a_n \psi_n \in \mathcal{A}_k$. Since f is linear and continuous, we have that

$$(f,\phi) = \sum_{n=0}^{\infty} \bar{a}_n b_n . \tag{2.1}$$

It follows from [6] that

$$\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} < \infty . \tag{2.2}$$

So, there exists an element $g \in \mathcal{A}_{-k}$ such that $g = \sum_{n=0}^{\infty} b_n \psi_n$. Conversely, if we have $g = \sum_{n=0}^{\infty} b_n \psi_n \in \mathcal{A}_{-k}$, such that relation (2.1) holds, then the mapping $\phi = \sum_{n=0}^{\infty} a_n \psi_n \to \sum_{n=0}^{\infty} \bar{a}_n b_n$, $\phi \in \mathcal{A}_{-k}$, defines an element from $\mathcal{A}'_k$. Denote this element by f. It is obvious that $b_n = (f,\psi_n)$, $n \in \mathbb{N}_0$. Hence, we have a one-to-one mapping $f \in \mathcal{A}'_k \to \sum_{n=0}^{\infty} b_n \psi_n \in \mathcal{A}_{-k}$, where $b_n = (f,\psi_n)$, $n \in \mathbb{N}_0$. Obviously, this mapping is linear.

Next, we shall prove that $|f|'_k = \|f\|_{-k}$, where $|f|'_k$ is the dual norm in $\mathcal{A}'_k$.

We have

$$|(f,\phi)| = \Big| \sum_{n=0}^{\infty} b_n \bar{a}_n \Big| \le \Big(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \Big)^{1/2} \Big(\sum_{n=0}^{\infty} |a_n|^2 \tilde{\lambda}_n^{2k} \Big)^{1/2}$$

$$|(f,\phi)| \le \|f\|_{-k} \|\phi\|_k$$

$$|f|' \le \|f\| \quad .$$

Furthermore let $\phi_m = \sum_{n=0}^{m} b_n \tilde{\lambda}_n^{-2k} \psi_n \in \mathcal{A}_k$. We have

$$\|\phi_m\|_k = \left[\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k}|^2 \; \tilde{\lambda}_n^{2k} \right]^{1/2} = \left[\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} \right]^{1/2},$$

so that

$$|f|_k' = \frac{(f,\phi_m)}{\|\phi_m\|_k} = \frac{\sum_{n=0}^{m} |b_n|^2 \tilde{\lambda}_n^{-2k}}{\left(\sum_{n=0}^{m} |b_n|^2 \tilde{\lambda}_n^{-2k}\right)^{1/2}} = \left(\sum_{n=0}^{m} |b_n|^2 \tilde{\lambda}_n^{-2k}\right)^{1/2} \xrightarrow[m\to\infty]{} \left(\sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k}\right)^{1/2} = \|f\|_{-k}.$$

It follows that $|f|_k' = \|f\|_{-k}$.

We shall write $X \hookrightarrow Y$ to denote that a topological vector space X can be embedded linearly and continuously into a topological vector space Y.

One can prove easily that

$$\dots \mathcal{A}_{k+1} \hookrightarrow \mathcal{A}_k \hookrightarrow \dots \hookrightarrow \mathcal{A}_1 \hookrightarrow \mathcal{A}_0 = L^2(I) \hookrightarrow \mathcal{A}_{-1} \dots \mathcal{A}_{-k} \hookrightarrow \dots$$

Let

$$\mathcal{A} = \bigcap_{k=0}^{\infty} \mathcal{A}_k = \left\{ \phi \in L^2(I): \quad \phi = \sum_{n=0}^{\infty} a_n \psi_n; \quad \forall k \sum_{n=0}^{\infty} |a_n|^2 \tilde{\lambda}_n^{2k} < \infty \right\}.$$

$$\mathcal{A}' = \bigcap_{k=0}^{\infty} \mathcal{A}_{-k} = \left\{ f: \quad f = \sum_{n=0}^{\infty} b_n \psi_n, \quad \exists k \sum_{n=0}^{\infty} |b_n|^2 \tilde{\lambda}_n^{-2k} < \infty \right\}.$$

Note that the space $\mathcal{A}$ is dense in $\mathcal{A}_k$, $k \in \mathbb{N}_0$, because it contains the set S which is dense in each $\mathcal{A}_k$, $k \in \mathbb{N}_0$. So, $\mathcal{A}_k$, $k \in \mathbb{N}_0$, is the completion of $\mathcal{A}$ with respect to the norm $\|\cdot\|_k$. From Theorem 2.1. it follows that $\mathcal{A}'$ is the dual of $\mathcal{A}$, 6. From 11, Lemma 9.3.3, p.316 it follows that the spaces $\mathcal{A}$ and $\mathcal{A}'$ are identical to the spaces defined in 11, ch.9.3. and 9.6. and denoted by the same letters.

Definition. *An element from* $\mathcal{A}_{-k}$ *(i.e.* $\mathcal{A}_k'$*) is called a generalized function of* $\mathcal{R}$*-order* k. *An element from* $\mathcal{A}'$ *is called a generalized function of* $\mathcal{R}$*-finite order.*

2.3. Let $m,k \in \mathbb{N}_0$. In Section 2.2. we defined the mappings $\mathcal{R}^m: \mathcal{A}_k \to L^2(I)$, $m \le k$. We define the mappings $(\mathcal{R}^m)': L^2(I) \to \mathcal{A}_{-k}$, $m \le k$, in the following way

(2.3) $((\mathcal{R}^m)'f,\phi) = (f,\mathcal{R}^m\phi),\ m \le k,\ \ \phi \in \mathcal{A}_k,\ \ f \in L^2(I)$.

If $f \in L^2(I)$ is the of form $f = \sum_{n=0}^{\infty} b_n\psi_n,\ \sum_{n=0}^{\infty} |b_n|^2 \le \infty$, we have

(2.4) $(\mathcal{R}^m)'f = \sum_{n=0}^{\infty} b_n\ \lambda_n^m\ \psi_n$.

It is obvious that $(\mathcal{R}^m)'$ can be defined on $\mathcal{A}_{-p}$, $p \in N$, in the same way as in (2.3) and that $(\mathcal{R}^m)': \mathcal{A}_{-p} \to \mathcal{A}_{-p-m}$. As it formally $(\mathcal{R}^m)'f = \sum_{n=0}^{\infty} b_n\lambda_n^m\psi_n$, $f \in \mathcal{A}_{-k}$, $m \le k$, we shall denote $(\mathcal{R}^m)'$ by $\mathcal{R}^m$, $m \le k$,

Put $\Lambda = \{n \in \mathbb{N}_0 : \lambda_n = 0\}$, and $\Lambda^c = \mathbb{N}_0\backslash\Lambda$. It is easy to prove the following representation theorem

Theorem 2.2. *Let* $f \in \mathcal{A}_{-k}$ *be of the form* $f = \sum_{n=0}^{\infty} b_n\psi_n$, *and* $F = \sum_{n\in\Lambda^c}(b_n\lambda_n^{-k})\psi_n$. *Then, we have that* $F \in L^2(I)$ *and*

$$f = \mathcal{R}^k F + \sum_{n\in\Lambda} b_n\psi_n\ .$$

So

$$f \in \mathcal{A}' \Leftrightarrow \exists k \in \mathbb{N}_0,\ \exists F \in L^2(I),\ \ \exists b_n \in \mathbb{C},\ n \in \Lambda,\ f = \mathcal{R}^k F + \sum_{n\in\Lambda} b_n\psi_n\ .$$

3. Generalized random processes from $L(\mathcal{A}_k, Z)$, $k \in \mathbb{N}_0$.

3.1. Let $(\Omega,\mathcal{F},P)$ be a probability space. Denote by Z the space of all the P-equivalence classes of complex random variables with finite second momenta. Z is the Hilbert space with the scalar product and the norm defined in the usual way. For $\xi,\ \eta \in Z$,

$$(\xi,\eta)_Z = E\xi\bar{\eta} = \int_\Omega \xi(\omega)\bar{\eta}(\omega)dP(\omega)\ ,$$

$$\|\xi\|_Z = (\xi,\xi)_Z^{1/2}.$$

We suppose that Z is separable, so there exists o.n.s $\{\xi_n,\ n \in \mathbb{N}_0\}$ and foe $\xi \in Z$ we have $\xi = \sum^{\infty} c_n\xi_n$, $c_n = (\xi,\xi_n)_Z$, $n \in \mathbb{N}_0$, $\sum^{\infty} |c_n|^2 < \infty$.

In the definition and assertion which are to follow in Section 3. and 4. we follow [4].

Definition. *Random process $\xi(t)$ on I is a family $\{\xi(t),\ t \in I\}$ of random variables from Z.*

Denote by $L(X,Y)$ a vector space of all the linear and contonuous mappings from a topological vector space X to a topological space Y.

Definiton. *Generalized random process on $\mathcal{A}$ is an element from $L(\mathcal{A},Z)$. Denote $\mathcal{A}^* = L(\mathcal{A},Z)$, and by (ξ,ϕ) the value of $\xi \in \mathcal{A}^*$ at $\phi \in \mathcal{A}$. A sequence $\{\xi_n, n \in \mathbb{N}\}$ converges to $\xi \in \mathcal{A}^*$ if for each $\phi \in \mathcal{A}$, $\lim_{n\to\infty} (\xi_n,\phi) = (\xi,\phi)$ in Z.*

We can define the operator $\mathcal{R}^k$ on the space $\mathcal{A}^*$ in the following way

$$(\mathcal{R}^k \xi,\phi) = (\xi,\mathcal{R}^k\phi), \quad k \in \mathbb{N}_0 .$$

Denote $\mathcal{A}_k^* = L(\mathcal{A}_k,Z)$. The norm on $\mathcal{A}_k^*$ is defined in the following way

$$\|\eta\|_{-k}^* = \sup \left\{ \|(\eta,\phi)\|_Z, \quad \phi \in \mathcal{A}\ ,\ \|\phi\| \le 1 \right\} .$$

The space $\mathcal{A}_k^*$ is completa because Z is complete. The relation $\mathcal{A}_k' = \mathcal{A}_{-k} \hookrightarrow \mathcal{A}_k^*$ holds, and for $f \in \mathcal{A}_{-k}$ we have $\|f\|_{-k}^* = \|f\|_{-k} = |f|_k'$. Since for $n \ge m \ge 0$ we have $\mathcal{A}_n \hookrightarrow \mathcal{A}_m$ and $\|\phi\|_m \le \|\phi\|_n$, it follows that $\mathcal{A}_m^* \hookrightarrow \mathcal{A}_n^*$. Also, $\mathcal{A} \hookrightarrow \mathcal{A}_k$, and every convergent sequence in $\mathcal{A}$ is convergent in $\mathcal{A}_k$, so $\mathcal{A}_k^* \hookrightarrow \mathcal{A}^*$. Therefore, the spaces $\mathcal{A}_k^*$ satisfy:

$$(L^2(I))^* = \mathcal{A}_0^* \hookrightarrow \mathcal{A}_1^* \ \dots\ \mathcal{A}_k^* \hookrightarrow \dots \hookrightarrow \mathcal{A}^*$$

and moreover

$$\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}_k^*, \quad \text{(in the set theoretical sense)}$$

Definition. *An element from $\mathcal{A}_k^* = L(\mathcal{A}_k,Z)$ is called the generalized random process of $\mathcal{R}$-order* k.

3.2. Denote by Y^∞ a space of random processes on I of the form

$$\eta(t,\omega) = \sum_{n=0}^{m} a_n(\omega)\psi_n(t)\ ,\quad a_n(\omega) \in Z,\quad t \in I\ .$$

Obviously $S \subset Y^\infty$.

Lemma 3.1. Y^{∞} *is a subspace of* $\mathcal{A}_k^*$, $k \in \mathbb{N}_0$.

Proof. Let $\eta \in Y^{\infty}$ be of the form $\eta = \sum_{n=0}^{m} a_n(\omega)\psi_n$. For $\phi \in \mathcal{A}_k$, $\phi = \sum_{n=0}^{m} b_n\psi_n$ we have

$$(\eta,\phi) = \left(\sum_{n=0}^{m} a_n(\omega)\psi_n, \phi\right) = \sum_{n=0}^{m} a_n(\omega)(\psi_n,\phi) = \sum_{n=0}^{m} a_n(\omega)\bar{b}_n ,$$

so $(\eta,\phi) \in Z$, for every $\phi \in \mathcal{A}_k$. Linearly is obvious.

To check continuity, let $\phi_n, \phi \in \mathcal{A}_k$, $n \in \mathbb{N}$ be of the form $\phi_n = \sum_{l=1}^{\infty} b_l^n\psi_l$, $\phi = \sum_{l=0}^{\infty} b_l\psi_l$, and let ϕ_n converge to ϕ in $\mathcal{A}_k$, i.e.

$$\sum_{l=0}^{\infty} |b_l^n - b_l|^2\, \tilde{\lambda}_l^{2k} \to 0, \quad n \to \infty$$

Therefore,

$$(\eta,\phi_n - \phi) = \sum_{l=0}^{m} a_l(\omega)(b_l^n - b_l) \to 0, \quad n \to \infty$$

in Z.

Denote by Y^{-k} the space obtained by completing Y^{∞} in $\mathcal{A}_k^*$ with respect to the norm $\|\cdot\|_{-k}^*$,

Lemma 3.2. $\mathcal{A}_{-k}$ *is a subspace of* Y^{-k} .

Proof. Let $f \in \mathcal{A}_{-k}$ be of the form $f = \sum_{n=0}^{\infty} a_n\psi_n$ and $\eta_m \in Y^{\infty}$ of the form $\eta_m = \sum_{n=0}^{m} a_n\psi_n$, $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$. We shall show that the sequence η_m convergence to f in Y^{-k}.

$$\|f - \eta_m\|_{-k}^* = |f - \eta_m|_k' = \|f - \eta_m\|_{-k} = \left(\sum_{n=m+1}^{\infty} |a_n|^2\tilde{\lambda}_n^{-2k}\right)^{1/2} \to 0, \quad m \to \infty$$

Theorem 3.1. Y^{-k} *is a proper subspace of* $\mathcal{A}_k^*$, $k \in \mathbb{N}_0$.

Proof. Let $\{\xi_n,\ n \in \mathbb{N}_0\}$ and $\{\tilde{\psi}_n,\ n \in \mathbb{N}_0\}$ be o.n.s. in Z and $\mathcal{A}_k$ respectively. Let $\phi = \sum_{n=0}^{\infty} c_n\psi_n = \sum_{n=0}^{\infty} c_n\tilde{\lambda}_m^k\tilde{\psi}_n$. Define an element $\eta^* \in L(\mathcal{A}_k, Z)$ by

$$(\eta^*,\phi) = \sum_{n=0}^{\infty} (\tilde{\psi}_n\phi)_k\ \xi_n . \tag{3.1}$$

The mapping is well defined since

$$\|(\eta^{*},\ \phi)\|_Z^2 = \sum_{n=0}^{\infty} |(\tilde{\psi},\phi)_k|^2 = \sum_{n=0}^{\infty} |\bar{c}_n|^2 \tilde{\lambda}_n^{2k} < \infty ,$$

where

$$(\tilde{\psi},\phi)_k = \left(\frac{\psi_n}{\tilde{\lambda}_n^k} , \sum_{n=0}^{\infty} c_n \psi_n \right)_k = \frac{\bar{c}_n\ \tilde{\lambda}_n^{2k}}{\tilde{\lambda}_n^k} = \bar{c}_n \tilde{\lambda}_n^k .$$

Linearly is obvious and if $\phi_m \to \phi$ in $\mathcal{A}_k$, then

$$\|(\eta^{*}, \phi_m - \phi)\|_Z^2 = \sum_{n=0}^{\infty} |c_n^m - c_n|^2\ \tilde{\lambda}_n^{2k} \to 0, \quad m \to \infty ,$$

so that η^{*} is continuous and $\eta^{*} \in \mathcal{A}_k^{*}$.

The mappings $\phi \mapsto (\tilde{\psi}_n, \phi)$ $n \in \mathbb{N}_0$, are linear and continuous, with norms equal to 1, so there exist $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$,(Theorem 2.1.) such that $\|f_n\|_{-k}=1$ and

$$(\tilde{\psi}_n, \phi) = (f_n, \phi) \ , \quad n \in \mathbb{N}_0,$$

and

$$(\eta^{*}, \phi) = \sum_{n=0}^{\infty} (f_n, \phi)\xi_n \ .$$

Let η be an arbitrary element from Y^{∞} of the form

$$\eta(t,\omega) = \sum_{n=0}^{m} d_n(\omega)\ \tilde{\psi}_n(t) \ .$$

For $t \in I$, fixed, we have

$$\eta(t,\omega) = \sum_{n=0}^{m} g_n(t)\ \xi_n(\omega) \ ,$$

where

$$g_n(t) = (\eta(t,\omega),\ \xi_n(\omega))_Z \ , \quad n \in \mathbb{N}_0.$$

We shall prove

$$\sum_{n=0}^{\infty} \int_I |g_n(t)|^2 dt < \infty \ . \tag{3.2}$$

We have that

$$\int_I \|\eta(t,\omega)\|_Z^2\ dt = \int_I \left[\int_\Omega |\sum_{n=0}^{m} d_n(\omega)\ \tilde{\psi}_n(t)|^2\ dP\ (\omega) \right] dt \le$$

$$\le \int_I \left\{ \int_\Omega \left[\sum_{i,j \le m} d_i(\omega)\ \tilde{\psi}_i(t)\ \bar{d}_j(\omega)\ \bar{\tilde{\psi}}_j(t) \right] dP(\omega) \right\} dt =$$

$$= \sum_{i,j\le m} \int_I \tilde{\varphi}_i(t)\,\bar{\tilde{\varphi}}_j(t)dt \int_\Omega d_i(\omega)\bar{d}_j(\omega)dP(\omega) =$$

$$= \sum_{n=0}^{m} \frac{1}{\tilde{\lambda}_n^{2k}} \|d_n(\omega)\|_Z^2 < \infty .$$

Furthermore, since

$$\int_I \sum_{n=0}^{\infty} |g_n(t)|^2 dt = \int_I \|\eta(t,\omega)\|_Z^2 dt < \infty ,$$

and according Lebesgue's theorem

$$\sum_{n=0}^{\infty} \int_I |g_n(t)|^2 dt = \int_I \sum_{n=0}^{\infty} |g_n(t)|^2 dt < \infty ,$$

(3.1) follows.

Hence, $\|g_n(t)\|_0^2 = \int_I |g_n(t)|^2 dt \to 0,\ n \to \infty$. Since $\|f_n\|_{-k}=1$ and $\|g_n\|_{-k} \le \|g_n\|_0 \to 0$, we have

$$\|\eta^* - \eta\|_{-k}^{*2} = \sup \{\|(\eta^* - \eta,\phi)\|_Z^2 ,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\} =$$

$$= \sup \{ \sum_{n=0}^{\infty} |(f_n - g_n,\phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\} \ge$$

$$\ge \lim_{n\to\infty} \sup \{|(f_n - g_n,\phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\} =$$

$$= \lim_{n\to\infty} \sup \{|(f_n,\phi)|^2 - 2|(f_n,\phi)|\cdot|(g_n,\phi)| + |(g_n,\phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\} \ge$$

$$\ge \lim_{n\to\infty} \{\|f_n\|_{-k}^2 - 2\|f_n\|_{-k}\|g_n\|_{-k}\} = 1.$$

Since Y^∞ is dense in Y^{-k}, we have that for any $\eta \in Y^{-k}$, $\|\eta^* - \eta\|_{-k} \ge 1$. It follows that the element η^*, defined in (3.1) does not belong to Y^{-k}, so Y^{-k} is the proper subspace of $\mathcal{A}_k^*$.

4. Structural theorems

4.1. Definiton. *Let* $\{f_n,\ n \in \mathbb{N}_0\}$ *and* $\{\theta_n,\ n \in \mathbb{N}_0\}$ *be sequences from* $\mathcal{A}'$ *and* Z *respectively.Then* $\sum_{n=0}^{\infty} f_n \otimes \theta_n$ *is a generalized random proces in* $\mathcal{A}^*$ *defined by*

$$\left[\sum_{n=0}^{\infty} f_n \otimes \theta_n , \phi \right] = \lim_{m\to\infty} \sum_{n=0}^{m} (f_n,\phi)\, \theta_n , \quad \forall\phi \in \mathcal{A} ,$$

provided that the limit on the right-hand side exists in Z *for each* $\phi \in \mathcal{A}$.

Linearity is obvious and continuity follows form the Banach Steinhaus theorem, so that our definition is correct.

Theorem 4.1. *Let* $\eta \in \mathcal{A}^*$ *and* $\{\xi_n,\ n \in \mathbb{N}_0\}$ *be o.n.s in Z. Then* η *belongs to* $\mathcal{A}_k^*$, $k \in \mathbb{N}_0$ *if and only if* η *is expressible in the form*

$$\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n \,, \tag{4.1}$$

where $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$ and for every $\phi \in \mathcal{A}_k$

$$\sum_{n=0}^{\infty} |(f_n \phi)|^2 < \infty \,. \tag{4.2}$$

Proof. Let $\eta \in \mathcal{A}_k^*$. Then the mappings $\phi \to ((\eta,\phi),\xi_n)_Z$ are in $\mathcal{A}_k' = \mathcal{A}_{-k}$ for every $\xi_n \in Z$, $n \in \mathbb{N}_0$. So, there exist $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$ such that

$$((\eta,\phi),\xi_n)_Z = (f_n,\phi), \quad \forall\phi \in \mathcal{A}_k, \quad n \in \mathbb{N}_0 \,.$$

and

$$(\eta,\phi) = \sum_{n=0}^{\infty} ((\eta,\phi),\xi_n)_Z \xi_n = \left[\sum_{n=0}^{\infty} (f_n \otimes \xi_n),\phi \right], \quad \forall\phi \in \mathcal{A}_k \,.$$

So (4.1) follows.

Further, we have

$$\infty > \|\eta,\phi)\|_Z^2 = \sum_{n=0}^{\infty} |((\eta,\phi),\xi_n)_Z|^2 = \sum_{n=0}^{\infty} |(f_n,\phi)|^2, \quad \forall\phi \in \mathcal{A}_k \,,$$

which proves (4.2).

Conversely, let $\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n$, $f_n \in \mathcal{A}_{-k}$, $\sum_{n=0}^{\infty} |(f_n,\phi)|^2 < \infty$, $\forall\phi \in \mathcal{A}_k$.

Since $\eta \in \mathcal{A}^*$ we have

$$(\eta,\phi) = \sum_{n=0}^{\infty} (f_n,\phi)\xi_n \in Z, \quad \forall\phi \in \mathcal{A}.$$

Consider the sequence

$$\eta_m = \sum_{n=0}^{m} f_n \otimes \xi_n .$$

It is easy to prove that η_m is a Cauchy sequence in $\mathcal{A}_k^*$. Namely, it is obvious that $\eta_m \in \mathcal{A}_k^*$, $m \in \mathbb{N}_0$, and for $\phi \in \mathcal{A}_k$ and $l > k$.

$$\|(\eta_l,\phi) - (\eta_k,\phi)\|_Z^2 = \sum_{n=k+1}^{l} |(f_n,\phi)|^2 < \varepsilon, \quad l,k > k_0(\varepsilon).$$

Since $\mathcal{A}_k^*$ is complete, the sequence η_m converges in $\mathcal{A}_k^*$ to an element $\eta_0 \in \mathcal{A}_k^*$.

Let $\eta_0 = \sum_{n=0}^{m} \tilde{f}_n \otimes \xi_n$. We shall show that $\eta_0 = \eta$.

Since $\eta \to \eta_0$ in $\mathcal{A}_k^*$, for every $n \in \mathbb{N}_0$, we have

$$0 = \lim_{m\to\infty} ((\eta_0 - \eta_m,\phi),\xi_n)_Z = \lim_{m\to\infty} \Big[((\eta_0,\phi),\xi_n)_Z - ((\eta_m,\phi),\xi_n)_Z\Big] =$$

$$= ((\eta_0,\phi),\xi_n)_Z - ((\eta,\phi),\xi_n)_Z = (\tilde{f}_n,\phi) - (f_n\phi).$$

So we have

$$((\tilde{f}_n,\phi) = (f_n,\phi), \quad \forall n \in \mathbb{N}_0 \text{ and } \forall\phi \in \mathcal{A}_k.$$

It follows that $\eta_0 = \eta$, $\eta_m \to \eta$ in $\mathcal{A}_k^*$ and η is of the form $\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n$.

Theorem 4.2. *Let $\eta \in \mathcal{A}^*$ and $k \in \mathbb{N}_0$. Then η belongs to Y^{-k} if and only if it can be represented in the form*

$$\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n . \tag{4.3}$$

where $f_n \in \mathcal{A}_{-k}$ and for every $\phi \in \mathcal{A}_k$, $\sum_{n=0}^{\infty} |(f_n,\phi)|^2 < \infty$, and the sequence

$$\eta_0 = \sum_{n=0}^{m} f_n \otimes \xi_n . \tag{4.4}$$

is a Cauchy sequence in Y^{-k}.

Proof. Let η be in Y^{-k}. From Theorem 3.1. $Y^{-k} \subset \mathcal{A}_k^*$, so according to Theorem 4.1. $\eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n$ where $f_n \in \mathcal{A}_{-k}$ and $\sum_{n=0}^{\infty} |(f_n,\phi)|^2 < \infty$ for every $\phi \in \mathcal{A}_k$. To prove that $\{\eta_m,\ m \in \mathbb{N}_0\}$ is the Cauchy sequence in Y^{-k} we shall show that $\eta_m \in Y^{-k}$, $m \in \mathbb{N}_0$, first. Since the set S is dense in $\mathcal{A}_{-k}$, it follows that for every $f_n \in \mathcal{A}_{-k}$, $n \in \mathbb{N}_0$, there exists a sequence $\{f_n^i,\ i \in \mathbb{N}_0\}$ in S such that $\|f_n^i - f_n\|_{-k} \to 0$, $i \to \infty$.

Define

$$\eta_m^i = \sum_{n=0}^{m} f_n^i\, \xi_n\ , \quad i \in \mathbb{N}_0\ ;\quad m \in \mathbb{N}_0.$$

For every $i, m \in \mathbb{N}_0$, η_m^i is in Y^{∞} because f_n^i, $n, i \in \mathbb{N}_0$ are in S.

Furthermore,

$$\|\eta_m - \eta_m^i\|_{-k}^{*2} = \Big\|\sum_{n=0}^{m}(f_n - f_n^i)\xi_n\Big\|_{-k}^{*2} = \sum_{n=0}^{m} \|f_n - f_n^i\|_{-k}^2 \to 0, \quad i \to \infty.$$

So, for every $m \in \mathbb{N}_0$, η_m is in Y^{-k}.

Next, we shall prove that $\{\eta_m,\ m \in \mathbb{N}_0\}$ is a Cauchy sequence in Y^{-k}. Since η is in Y^{-k}, there exists a sequence $\{\theta_j,\ j \in \mathbb{N}_0\}$ in Y^{-k} such that $\|\eta - \theta_j\|_{-k}^{*} \to 0$, $j \to \infty$. Each θ_j can be represented in the form $\theta_j = \sum_{n=0}^{\infty} f_n^j\, \xi_n$ where $f_n^j = (\theta_j, \xi_n)_Z$ are all in S. Since $\{\xi_n,\ n \in \mathbb{N}_0\}$ is the complete o.n.s. in Z and θ_j is in Y^{∞} we have

$$\int_I \|\theta_j\|_Z^2\, dt = \sum_{n=0}^{\infty} \|f_n^j\|_0^2. \tag{4.5}$$

For arbitrary m and j there holds

$$\|\eta - \eta_m\|_{-k}^{*} \le \|\eta - \theta_j\|_{-k}^{*} + \|\theta_j - \eta_m\|_{-k}^{*}\ .$$

We have

$$\|\theta_j - \eta\|_{-k}^{*2} = \sup\Big\{ \sum_{n=0}^{\infty} |(f_n^j - f_n, \phi)|^2,\ \ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\Big\}, \tag{4.6}$$

$$\|\theta_j - \eta_m\|_{-k}^{*2} = \sup\Big\{ \sum_{n=0}^{m} |(f_n^j - f_n, \phi)|^2 + \sum_{n=m+1}^{\infty} |(f_n^j, \phi)|^2,\ \ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1\Big\}, \tag{4.7}$$

$$\le \sup\Big\{ \sum_{n=0}^{m} |(f_n^j - f_n, \phi)|^2,\ \ \phi \in \mathcal{A}_k,\ \ \|\phi\|_k \le 1 \Big\} +$$

$$+ \sup\Big\{ \sum_{n=m+1}^{\infty} |(f_n^j, \phi)|^2,\ \ \phi \in L^2(I),\ \|\phi\|_0 \le 1\Big\} \le$$

$$\le \sup\left\{ \sum_{n=0}^{\infty} |(f_n^j - f_n, \phi)|^2,\ \phi \in \mathcal{A}_k,\ \|\phi\|_k \le 1 \right\} + \sum_{n=m+1}^{\infty} \|f_n^j\|_0^2 =$$

$$= \|\theta_j - \eta\|_{-k}^{*2} + \sum_{n=m+1}^{\infty} \|f_n^j\|_0^2.$$

So, from (4.6) and (4.7) it follows that

$$\|\eta - \eta_m\|_{-k}^{*} \le \|\eta - \theta_j\|_{-k}^{*} + \left\{ \|\eta - \theta_j\|_{-k}^{*2} + \sum_{n=m+1}^{\infty} \|f_n^j\|_0^2 \right\}^{1/2}$$

Since $\theta_j \to \eta$ in Y^{-k}, for arbitrary $\varepsilon > 0$ there exist $j_0 = j_0(\varepsilon)$ such that $\|\eta - \theta_j\|_{-k}^{*} \le \varepsilon/3$ for every $j \ge j_0$.

Furthermore, from (4.5) it follows that there exists $m_0 = m_0(j)$ such that $\sum_{n=m+1} \|f_n^j\|_0^2 < 3\varepsilon^2/9$, for all $m \ge m_0$. Hence, for every $j \ge j_0$ and $m \ge m_0$

$$\|\eta - \eta_m\|_{-k}^{*} \le \frac{\varepsilon}{3} + \left\{ \frac{\varepsilon^2}{9} + \frac{3\varepsilon^2}{9} \right\}^{1/2} = \varepsilon .$$

Since ε was arbitrary, it follows that the sequence $\{\eta_m,\ m \in \mathbb{N}_0\}$ converges to η in Y^{-k}, which means that it is the Cauchy sequence.

Conversly, let η and η_m be defined as in (4.3), (4.4). We shall show that η is in Y^{-k}. Since $\{\eta_m,\ m \in \mathbb{N}_0\}$ is a Cauchy sequence in Y^{-k}, it converges to an element; denote it by η_0, in Y^{-k}. We have that

$$\eta_0 = \sum_{n=0}^{\infty} f_n^0 \otimes \xi_n, \quad \eta = \sum_{n=0}^{\infty} f_n \otimes \xi_n \quad \text{and} \quad \eta_m = \sum_{n=0}^{m} f_n \otimes \xi_n ,$$

so,

$$0 = \lim_{m\to\infty} ((\eta_0 - \eta, \phi), \xi_n)_Z = \lim_{m\to\infty} \left[((\eta_0 - \phi), \xi_n)_Z - ((\eta_m - \phi), \xi_n)_Z \right] =$$

$$= ((\eta_0 - \phi), \xi_n)_Z - ((\eta_m - \phi), \xi_n)_Z = (f_n^0, \phi) - (f_n, \phi), \quad \forall n \in \mathbb{N}_0, \quad f \in \mathcal{A}_k.$$

We have $(f_n^0, \phi) = (f_n, \phi)$, $\forall n \in \mathbb{N}_0$, $\forall \phi \in \mathcal{A}_k$, i.e. $\eta = \eta_0$, which means η is in Y^{-k}.

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Rezime

JEDNA KLASA UOPŠTENIH SLUČAJNIH PROCESA SA VREDNOSTIMA U $L^2(\Omega)$

Date su strukturne teoreme za uopštene slučajne procese koji pripadaju prostorima $L(\mathcal{A}_k, Z)$, Y^{-k}.

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