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EXAMPLE OF A CONTINUOUS NON-MARKOVIAN PROCESS $X(t)=W_1(t)+\varphi(t)W_2(t)$ OF MULTIPLICITY TWO

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Abstract

The continuous process $X(t)=W_1(t)+\varphi(t)W_2(t)$, $0\le t\le 1$, is considered, where $W_1(t)$ and $W_2(t)$ are independent Wiener processes and $\varphi(t)$ is a version of the Cantor distribution of function. The multiplicity of the non-Markovian process $X(t)$ is two. The proper canonical representation of $X(t)$ is also given.

0. Let $\{X(t),\ t\ge 0\}$ be a mean-square continuous and purely non-deterministic process. The proper canonical (or Hida-Cramer) representation of $\{X(t)\}$ is

$$X(t)=\sum_{1}^{N}\int_{0}^{t} g_n(t,u)\ dZ_n(u)\ , \tag{1}$$

where

- the so-called innovation processes $\{Z_n(t),t\ge 0\}$, $n=1,\dots,N$ (N may be ∞) are mutually orthogonal wide-sense martingals,
- the mean-square linear closure of $\{X(u),\ u\le t\}$: $\mathcal{H}(X;t)$ coincides with $\sum_{1}^{N}\oplus\ \mathcal{H}(Z_n;t)$ for all t ($\mathcal{H}(X)=\overline{\bigvee_t \mathcal{H}(X;t)}$ is the Hilbert space with the inner product $\langle X,Y\rangle = EXY$),

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- the measures $dF_n(t) = \|dZ_n(t)\|^2$ are ordered by the absolute continuity

$$dF_1(t) \gtrsim dF_2(t) \gtrsim \dots \gtrsim dF_N(t) .$$

Let ρ_n be the class of measures equivalent by the absolute continuity to $dF_n(t)$. The chain

$$\rho_1 \gtrsim \rho_2 \gtrsim \cdots \gtrsim \rho_N \tag{2}$$

is the spectra type of $\{X(t)\}$ and N is the spectral multiplicity of $\{X(t)\}$. The correlation function of $\{X(t)\}$ uniquely determines the spectral type. The main result of [1] is that for an arbitrary chain (2) there exists a continuous process with the spectral type (2).

In connection with this result interesting are the examples of univariate continuous processes with $N \geq 2$. It was shown in [3] that the M-ple Markovian process $\{Y(t)\}$ with the proper Goursat representation

$$Y(t) = \sum_1^M f_n(t)\ Z_n(t) \qquad ;$$

has the multiplicity $N \in \{1,\dots,M\}$ depending of some continuity and smoothness properties of the functions $f_n(t)$. For example, ([3], [2]) the process $\{Y(t)\}$ with the representation

$$Y(t) = W_1(t) + f(t)W_2(t) , \tag{3}$$

where $\{W_1(t)\}$ and $\{W_2(t)\}$ are mutually orthogonal Wiener processes, has the spectral type

$$dt \sim dt ,$$

if $f(t)$ is continuous but not absolutely continuous in any interval. In this case, $\{Y(t)\}$ is the 2-ple Markovian process and (3) is its proper canonical representation.

1. In this note we shall give the example of a continuous non-Markovian process $\{X(t),\ 0 \leq t \leq 1\}$ with the representation

$$X(t) = W_1(t) + \varphi(t)\ W_2(t) , \tag{4}$$

where $\{W_1(t)\}$ and $\{W_2(t)\}$ are multually orthogonal Wiener processes. The function $\varphi(t)$ is a modification of the distribution function on the Cantor ternary set.

We define $\varphi(t)$, $0 \le t \le 1$, in the following way: $\varphi(0) = 0$, $\varphi(1) = 1$;
$\varphi(t) = \frac{2}{3}\varphi(0) + \frac{1}{3}\varphi(1) = \frac{1}{3}$, $\frac{1}{3} \le t \le \frac{2}{3}$; $\varphi(t) = \frac{2}{3}\varphi(0) + \frac{1}{3}\varphi(\frac{1}{3}) = \frac{1}{9}$, $\frac{1}{9} \le t \le \frac{2}{9}$;
$\varphi(t) = \frac{2}{3}(\frac{2}{3}) + \frac{1}{3}\varphi(1) = \frac{5}{9}$, $\frac{7}{9} \le t \le \frac{8}{9}$; ... and so on.

In such a way $\varphi(t)$ is a continuous distribution function increasing only on the Cantor set C. Let us denote by L and R; $L, R \subset C$; the countable sets of left and right, respectively, end-points of removed open intervals in the definiton of C. According to the definition of $\varphi(t)$, it is easy to see that for $t' \in L$

(5) $$\lim_{h \downarrow 0} \frac{[\varphi(t') - \varphi(t'-h)]^2}{h} = +\infty$$

In such a way, for the process $\{X(t)\}$ defined by (4), we have

(6) $$\underset{h \downarrow 0}{\text{l.i.m.}} \frac{X(t') - X(t'-h)}{\varphi(t') - \varphi(t'-h)} = W_2(t') = \dot{X}_\varphi(t') \ .$$

Indeed, $(\Delta\varphi = \varphi(t') - \varphi(t'-h))$,

$$\left\| \frac{X(t') - X(t'-h)}{\Delta\varphi} - W_2(t') \right\|^2 = \frac{1}{(\Delta\varphi)^2} \| W_1(t') - W_1(t'-h) +$$

$$+ \varphi(t'-h)[W_2(t') - W_2(t'-h)] \|^2 = \frac{1}{(\Delta\varphi)^2} [1 + \varphi^2(t'-h)] \longrightarrow 0 \quad h \downarrow 0.$$

Let us denote by P_t the projection operator onto $\mathcal{H}(X, t)$ and consider the wide-sense martingal $\{Z_1(t),\ 0 \le t \le 1\}$ defined by

$$Z_1(t) = P_t X(1).$$

Evidently, $\{Z_1(t)\}$ is an innovation process of $\{X(t)\}$.

$\{X(t)\}$ being continuous, the space $\mathcal{H}(X)$ is separable. For arbitrary t, $0 < t < 1$, consider the partition $\frac{k}{2^n} t$, $k = 1, \ldots, 2^n$ of $[0, t]$ for $n = 1, 2, \ldots$ and the mean-square linear closure $\mathcal{H}_n(X; t)$ of $\{X(\frac{k}{2^n} t),\ k = 1, \ldots, 2^n\}$. By the separability of $\mathcal{H}(X; t)$ and $\mathcal{H}_1(X; t) \subset \mathcal{H}_2(X; t) \subset \ldots$ we conclude that

$$\mathcal{H}(X; t) = \overline{\bigvee_n \mathcal{H}_n(X; t)} \ .$$

Let us denote by P_{nt} the projection operator onto $\mathcal{H}_n(X;t)$. We have

$$Z_1(t) = \underset{n\to\infty}{\text{l.i.m.}}\ P_{nt}X(t)\ . \tag{7}$$

Simplifyng the notations by $h = \frac{1}{2^n}t$, we consider the projection of $X(1)$ on random variables $X(t)$, $X(t-h)$, $X(t-2h)$, ...:

$$P_{nt}X(1) = aX(t) + bX(t-h) + cX(t-2h) + \ldots$$

It is easy to see that

$$\langle X(1) - [aX(t) + bX(t-h)],\ X(u)\rangle = 0 \quad \text{for all} \quad u \le t - h\ ,$$

if $\quad a = \dfrac{1 - \varphi(t - h)}{\Delta\varphi}\ , \quad b = \dfrac{\varphi(t) - 1}{\Delta\varphi}\ .$

Rewrite

$$aX(t) + bX(t - h) = X(t) + [1 - \varphi(t)]\ \frac{X(t) - X(t - h)}{\Delta\varphi}\ .$$

By (5), we have for $t'\in L$

$$Z_1(t') = X(t') + [1 - \varphi(t')]\dot{X}_{\varphi}(t') = W_1(t') + W_2(t') \tag{8}$$

Put

$$Z_2(t') = W_1(t') - W_2(t')\ . \tag{9}$$

We have $\langle Z_1(t'),\ Z_2(t')\rangle = 0$. Using (6) and (7), we write

$$X(t') = \frac{1 + \varphi(t')}{2}\, Z_1(t') + \frac{1 - \varphi(t')}{2}\, Z_2(t')\ . \tag{10}$$

We conclude from (6) that $W_2(t') \in \mathcal{H}(X;t')$ and from (8) and (9) we conclude that $Z_1(t')$, $W_1(t')$, $Z_2(t') \in \mathcal{H}(X;t')$. In this way (10) is the proper canonical representation of $\{X(t)\}$ at the points $t'\in L$.

Let (t',t'') be a removed interval and let $t \in (t',t'']$. We verify, by simple calculation, that

$$\left\langle X(1) - \left[Z_1(t') + \frac{1 + \varphi(t')}{1 + \varphi^2(t')}\,(X(t) - X(t'))\right],\ X(u)\right\rangle = 0 \quad \text{for all} \quad u \le t.$$

This way

$$Z_1(t) = Z_1(t') + \alpha(t')(X(t) - X(t')),\quad \alpha(t') = \frac{1 + \varphi(t')}{1 + \varphi^2(t')}\ , \tag{11}$$

for $t \in [t',t'']$ with $Z_1(t'+0) = Z_1(t')$ and $Z_1(t''-0) = Z_1(t'')$.

Let us remark that $Z_1(t''+0) \neq Z_1(t'')$. Indeed, consider the sequence of intervals (t'_n, t''_n) such that $t'_n \downarrow t''$, $n \to \infty$. Then $Z_1(t'_n) = W_1(t'_n) + W_2(t'_n) \to$ $\to W_1(t'') + W_2(t'')$, in the mean-square as $n \to \infty$ and

$$\lim_{n\to\infty} \|Z_1(t'_n) - Z_1(t'')\|^2 = \| [W_1(t'') + W_2(t'')] - [Z_1(t') + \alpha(t')\, X(t'') - X(t')] \|^2 =$$

$$\frac{[1 - \varphi(t')]^2}{1 + \varphi^2(t')} (t'' - t') \neq 0$$

Rewrite (11) in the form

$$X(t) = X(t') + \frac{1}{\alpha(t')} (Z_1(t) - Z_1(t'))$$

or

$$(12) \quad X(t) = \frac{1 + \varphi^2(t')}{1 + \varphi(t')} Z_1(t) - \frac{(1 - \varphi(t'))^2}{2(1 + \varphi(t'))} Z_1(t') + \frac{1 - \varphi(t')}{2} Z_2(t').$$

(12) is the proper canonical representation of $\{X(t)\}$ for $t \in (t', t'']$.

There remains to find the proper canonical representation of $\{X(t)\}$ for $t = s \in C\setminus(L \vee R)$. Consider two sequences of removed intervals: the sequence $(\underline{t}'_n, \underline{t}''_n)$ such that $\underline{t}''_n \uparrow s$ (then in fact $\underline{t}'_n \uparrow s$, because $\underline{t}''_n - \underline{t}'_n \to 0$) and the sequence $(\bar{t}'_n, \bar{t}''_n)$ such that $\bar{t}'_n \downarrow s$. It follows from the continuity of $\{X(t)\}$ and $\{W_1(t)\}$ that

$$\underset{n\to\infty}{\text{l.i.m.}}\ X(\underline{t}'_n) = X(s) = \underset{n\to\infty}{\text{l.i.m.}}\ \bar{X}(t'_n)$$

or

$$\underset{n\to\infty}{\text{l.i.m.}} \left[\frac{1 + \varphi(\underline{t}'_n)}{2} Z_1(\underline{t}'_n) + \frac{1 - \varphi(\underline{t}'_n)}{2} Z_2(\underline{t}'_n) \right] = X(s) =$$

$$= \underset{n\to\infty}{\text{l.i.m.}} \left[\frac{1 + \varphi(\bar{t}'_n)}{2} Z_1(\bar{t}'_n) + \frac{1 - \varphi(\bar{t}'_n)}{2} Z_2(\bar{t}'_n) \right]$$

and

$$\|Z_1(\bar{t}'_n) - Z_1(\underline{t}'_n)\|^2 = \| [W_1(\bar{t}'_n) \pm W_2(\bar{t}'_n)] - [W_1(\underline{t}'_n) \pm W_2(\underline{t}'_n)] \|^2 =$$

$$= 2(\bar{t}'_n - \underline{t}_n) \to 0, \quad n \to \infty .$$

We conclude that

$$X(s) = \frac{s + \varphi(s)}{2} Z_1(s) + \frac{1 - \varphi(s)}{2} Z_2(s), \quad Z_i(s) = W_1(s) \pm W_2(s), \quad i=1,2,$$

is the proper canonical representation of $\{X(t)\}$.

Let us summarize:

The process $\{X(t),\ 0 \le t \le 1\}$ *given by* (4) *has the following proper canonical representation*

$$X(t) = \begin{cases} \dfrac{1+\varphi(t)}{2} Z_1(t) + \dfrac{1-\varphi(t)}{2} Z_2(t), & t \in C\setminus R \\[2ex] \dfrac{1+\varphi^2(t)}{1+\varphi(t)} Z_1(t) - \dfrac{(1-\varphi(t'))^2}{2(1+\varphi(t'))} Z_1(t') + \dfrac{1-\varphi(t')}{2} Z_2(t'), & t \in C^c \vee R \end{cases}$$

where $t' = t'(t)$ *is the left-end point of the removed interval* (t', t''), $t \in (t', t'']$.

The innovation processes $\{Z_i(t),\ 0 \le t \le 1\}$ $i=1,2$ *are*

$$Z_1(t) = \begin{cases} W_1(t) + W_2(t), & t \in C\setminus R \\[2ex] \dfrac{1+\varphi(t)}{1+\varphi^2(t)}[W_1(t) + \varphi(t)W_2(t)] + \dfrac{\varphi(t')(1-\varphi(t'))}{1+\varphi^2(t')} W_1(t') + \dfrac{1-\varphi(t')}{1+\varphi^2(t')} W_2(t'), \\ t \in C^c \vee R, \end{cases}$$

$$Z_2(t) = \begin{cases} W_1(t) - W_2(t), & t \in C\setminus R \\ W_1(t') - W_2(t'), & t \in C^c \vee R. \end{cases}$$

Let us make two remarks at the end.

The process $\{X(t)\}$ is not Markovian. According to the definition in [3], a process $\{Y(t)\}$ is M-ple Markovian if for all $a \le b$ set $\{P_a Y(t), t \ge b\}$ contains exactly M linearly independent elements. In our example, the set $\{P_a X(t),\ t \ge a\}$ for $a = t'$ contains two linearly independent elements $W_1(t'), W_2(t')$ and for $a = t \in (t', t'']$ contains three linearly independent elements $\dot{W}_1(t), W_1(t'), W_2(t')$.

Concerning the spectral type

$$\rho_1 > \rho_2$$

of the process $\{X(t)\}$, the measure $dF_2(t) = d\|Z_2(t)\|^2$ belonging to ρ_2 is a discrete measure on the set R, the measure $dF_1(t) = d\|Z_1(t)\|^2$ belonging to ρ_1 is the sum of a measure equivalent to the ordinary Lebesgue measure dt and a measure equivalent to $dF_2(t)$.

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Rezime

PRIMER NEPREKIDNOG NE-MARKOVSKOG PROCESA $X(t) = W_1(t) + \varphi(t)W_2(t)$ MULTIPLICITETA DVA

Posmatra se neprekidni proces $X(t) = W_1(t) + \varphi(t)W_2(t)$, $0 \leq t \leq 1$, gde su $W_1(t)$ i $W_2(t)$ nezavisni Vinerovi procesi i $\varphi(t)$ je jedna varijanta Kantorove funkcije raspodele. Multiplicitet ne-markovskog procesa $X(t)$ je dva. Takođe je data čisto kanonička reprezentacija za $X(t)$.

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