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MESH GENERATION METHODS FOR NUMERICAL SOLUTION OF QUASILINEAR SINGULAR PERTURBATION PROBLEMS

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Abstract

Quasilinear singularly perturbed boundary value problems are solved numerically by using finite - difference schemes on special non-equidistant meshes. The meshes are generated by suitable functions which redistribute equidistant points. Two similar approaches to mesh generation are compared. "Uniform" convergence is proved for two special types of problems. Numerical results illustrate efficiency of the methods.

1. Introduction

We consider the following singularly perturbed boundary value problem:

$$Tu := -\varepsilon u'' + b(u)u' + c(x,u) = 0\,, \quad x \in I = [0,1]\,, \tag{1.1}$$

$$Ru := (u(0), u(1)) = (U_0, U_1)\,, \tag{1.2}$$

where $' = d/dx$ and ε is a small parameter, $\varepsilon \in (0,1]$ (usually $\varepsilon \ll 1$). We assume:

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(1.3) $$b \in C^1(R)\ ,\quad c \in C^1(I \times R)\ ,$$

(1.4) $$c_u(x,u) \geq c_* > 0\ ,\quad x \in I\ ,\quad u \in R\ .$$

Then there exist numbers $\bar{u}$, $\underline{u}$, such that $\bar{u} > \underline{u}$ and:

(1.5) $$c(x,\underline{u}) \leq 0 \leq c(x,\bar{u})\ ,\quad x \in I\ ,$$

(1.6) $$\underline{u} \leq U_j \leq \bar{u}\ ,\quad j=0,1\ ,$$

and since the operator (T,R) is inverse monotone, there exists a unique solution u_ε, to the problem (1.1), (1.2), see [10], [11]. Moreover, for $x \in I$ we have

$$u_\varepsilon(x) \in W := [\underline{u},\bar{u}]\ ,$$

and

$$u_\varepsilon \in C^3(I)\ .$$

It is well known (see [4], [14], for instance) that u_ε may have one or more boundary or/and interior layers. Our aim is to solve (1.1), (1.2) numerically by using finite-difference schemes on special non-equidistant meshes. The meshes should be dense in the layers. On one hand this will give us a high percentage of numerical values in the regions where u_ε changes abruptly. On the other hand, we can prove a sort of uniform convergence (i.e. convergence uniform in ε) of the numerical solution towards u_ε, provided some additional information about bahaviour of u_ε be available.

Let us give some further details about our results and let us introduce some notation. First of all, we shall use (1.1) in the conservation form:

(1.7) $$Tu = -\varepsilon u'' + f(u)' + c(x,u) = 0$$

where

$$f_u(u) = b(u)\ .$$

We shall consider two different, but similar, ways of introducing non--equidistant meshes. We refer to them as *direct* and *indirect* approaches. In the direct approach we discretize (1.7) on some *non-equidistant* mesh I_h with the mesh points:

$$0 = x_0 < x_1 < \ldots < x_n = 1\ ,\quad n \in N\ .$$

We shall consider meshes generated by suitable functions λ, i.e.:

(1.8) $$x_i = \lambda(t_i)\ ,\quad t_i = ih\ ,\quad i=0,1,\dots,n\ ,$$

where $h=1/n$ and λ is sufficiently smooth function (at least from $C^1(I)$), which usually depends on ε and satisfies

(1.9) $$\lambda'(t) > 0\ ,\quad t \in I\ ;\quad \lambda(s)=s\ ,\quad s=0,1\ .$$

In the indirect approach we introduce a new independent variable t by $x=\lambda(t)$ and transform (1.7). Then the transformed problem is discretized on *equidistant* mesh $\tilde{I}_h$ with mesh points t_i. We use the notation (1.8) in this approach as well and assume that λ has the same properties as above.

Let

$$h_i = x_i - x_{i-1}\ ,\quad i=1,2,\dots,n\ ,$$

$$\bar{h}_i = (h_i + h_{i+1})/2\ ,\quad i=1,2,\dots,n-1,$$

and let w_h, v_h, etc. denote arbitrary mesh functions on $I_h \setminus \{0,1\}$ (or $\tilde{I}_h \setminus \{0,1\}$) which will be identified with R^{n-1} - column vectors. Thus,

$$w_h = [w_1, w_2, \dots w_{n-1}]^T\ ,\ (w_i := w_{h,i}\ ,\quad i=1,2,\dots,n-1).$$

By $w_{\varepsilon,h}$ we shall denote the numerical solution with components $w_{\varepsilon,i}$ which approximate $u_\varepsilon(x_i)$. Let

$$u_{\varepsilon,h} = [u_\varepsilon(x_1),\ u_\varepsilon(x_2),\dots,u_\varepsilon(x_{n-1})]^T\ .$$

Let $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote usual vector (matrix) norms in $R^{n-1}(R^{n-1,n-1})$. Define $\|\cdot\|_h$:

$$\|w_h\|_h = \sum_{i=1}^{n-1} \bar{h}_i\,|w_i| = \|Hw_h\|_1\ ,$$

where $H = \mathrm{diag}(\bar{h}_1,\ \bar{h}_2,\dots,\ \bar{h}_{n-1})$. The norm $\|\cdot\|$ is the standard L^1 discrete norm on non-equidestant meshes, cf. [1]. The corresponding matrix norm is:

(1.10) $$\|A\|_h = \|HAH^{-1}\|_1\ ,$$

where $A \in R^{n-1,n-1}$.

In both approaches we prove

(1.11) $$\|w_{\varepsilon,h} - u_{\varepsilon,h}\|_h \le M_\varepsilon h ,$$

where M_ε stands for any positive constant which is independent of h (but may depend on ε). We can prove (1.11) easily since we use stable discrete operators (Engquist - Osher and Lax - Friedrichs schemes, in particular). For instance, in the direct approach we prove the following stability inequality:

(1.12) $$\|w_h - v_h\|_h \le c_*^{-1}\|T_h w_h - T_h v_h\|_h ,$$

where T_h is a discrete operator corresponding to T from (1.7), $T_h: R^{n-1} \to R^{n-1}$. A similar result holds in the case of the indirect approach, as well.

The inequality (1.11) shows the linear discrete L^1 convergence of the numerical solution to the exact solution. The convergence is by no means uniform in ε: a constant M_ε is involved and even the norm $\|\cdot\|_h$ depends on ε if h_i's depend on it (which is the case when the mesh is dense in the layers for all values of ε). We can't avoid both causes of the non - uniformity and our aim is to use a special mesh (i.e. a special function λ) to get

(1.13) $$\|w_{\varepsilon,h} - u_{\varepsilon,h}\|_h \le Mh ,$$

where by M we denote any positive constant which is independent of h and ε. Although we use $\|\cdot\|_h$ which depends on ε, we shall refer to (1.13) as to the uniform convergence. Of course, it is not easy to obtain (1.13). We have to prove that the norm $\|\cdot\|_h$ of the consistency error of the discrete operator is bounded by Mh. This can be done only if we have sufficiently sharp estimates of the derivatives of u_ε (which occur in the consistency error) and if we use an appropriate function λ which condenses the mesh in the layers. Essentially, any part of λ which maps the mesh points into a layer behaves like inverse function of the corresponding boundary/interior layer function. Then, such parts are smoothly connected by appropriate polynomials.

Let us now introduce some more notation. Let

$$e_h = [1, 1, \dots, 1]^T \in R^{n-1} ,$$

$$W_h = \left\{ w_h \in R^{n-1} \mid \underline{u}e_h \le w_h \le \bar{u}e_h \right\} .$$

The inequality sign in R^{n-1} should be understood componentwise.

Finally, let g: $R^2 \to R$, $g \in C^1(R^2)$. The following properties of g will be of interest:

$$(1.14) \qquad g_v(u,v) \ge 0 \ge g_u(u,v) \;, \quad u,v \in R \;,$$

$$(1.15) \qquad g(u,u) = f(u) \;, \quad u \in W \;,$$

$$(1.16) \qquad |g_u(u_1,v) - g_u(u_2,v)| \le G|u_1 - u_2| \;, \quad u_1,u_2,v \in W \;,$$

$$(1.17) \qquad |g_v(u,v_1) - g_v(u,v_2)| \le G|v_1 - v_2| \;, \quad u,v_1,v_2 \in W \;,$$

with some $G \ge 0$. The function g will be used for discretization of the transport term in (1.7), e.g. in the direct approach:

$$f(u)'(x_i) \approx \Big[g(u(x_{i+1}),\ u(x_i)) - g(u(x_i),\ u(x_{i-1}))\Big]/\bar{h}_i \;,$$

cf. [1], [11-13].

The paper consists of 5 sections. After the Introduction, in Section 2 we consider the approach of direct discretization. First we prove the stability (1.12). This is a well known result for the class of schemes which we consider (cf. [1]), but we give different and simpler proof which uses the techniques of M - functions, see [3], [11-13]. Then we consider the consistency error and derive conditions on λ which imply (1.13). However, from these conditions it is not easy to see how to choose λ. The indirect approach, which we consider in Section 3, gives more information about the choice of λ. Section 3 contains the analysis of stability and consistency, similar to Section 2. In Section 4 we consider two special types of problem (1.1), (1.2), (a non-turning point problem and a boundary shock problem), which enables us to prove (1.13). A closer attention is kept on the direct approach, since these problems have already been solved numerically in [23] and [24] by the indirect approach. Finally, in Section 5 we present some numerical results and give some concluding remaks.

The mesh generation methods which we consider here date from 1969, [2]. In that paper the direct approach was introduced. The indirect approach was considered in [8] and [9], for instance. Other papers which use generated meshes are [7], [17-21], [25]. All of these papers deal with different types of linear or semilinear singular perturbation problems.

In general, mesh construction methods can be devided into implicit and explicit methods. In the explicit methods the mesh is given in advance. Thus, the methods which we use here belong to this class. Another explicit method is the method of Ascher and Weiss, see [26] for instance. The implicit methods, on the other hand, give as final numerical result both the numerical solution and the mesh points. To this class belong simultaneous methods, see [27], and alternate methods [6], [16].

2. The approach of direct discretization

Consider the following finite-difference operators on the mesh I_h :

$$D''_h z_i = (h_{i+1} z_{i-1} - 2\bar{h}_i z_i + h_i z_{i+1})/(h_i h_{i+1} \bar{h}_i) ,$$

$$D'_h z_i = \Big[g(z_{i+1}, z_i) - g(z_i, z_{i-1})\Big]/\bar{h}_i ,$$

where $\{z_i\}$ is a mesh function on I_h and g has been introduced is Section 1. Let T_h, T_h: $R^{n-1} \to R^{n-1}$, be given by:

$$T_h w_i := (T_h w_h)_i = -\varepsilon D''_h w_i + D'_h w_i + c(x_i, w_i) , \quad i=1,2,\dots,n-1 ,$$

where w_0 and w_n should be replaced by U_0 and U_1, respectively. Then the discretization of the problem (1.7), (1.2) reads:

(2.1) $$T_h w_h = 0 .$$

Theorem 2.1 *Let* (1.3), *(without* b_u, $c_x \in C(R)$), (1.4) *and* (1.14) *hold. Then there exists a unique solution* $w_{\varepsilon,h}$ *to the problem* (2.1), *and* $w_{\varepsilon,h} \in W_h$. *Moreover, for any* w_h, v_h *the stability inequality* (1.12) *holds.*

Proof. Let $T'_h(w_h)$ denote the Frechet derivative of the operator T_h. The i-th column ($i=1,2,\dots,n-1$) of $T'_h(w_h)$ has the following non-zero elements:

$$T'_h(w_h)_{i-1,i} = -\varepsilon/(h_i \bar{h}_{i-1}) + g_u(w_h, w_{i-1})/\bar{h}_{i-1} ,$$

$$T'_h(w_h)_{i,i} = 2\varepsilon/(h_i h_{i+1}) + \Big[g_v(w_{i+1}, w_i) - g_u(w_i, w_{i-1})\Big]/\bar{h}_i + c_u(x_i, w_i) ,$$

$$T'_h(w_h)_{i+1,i} = -\varepsilon/(h_{i+1} \bar{h}_{i+1}) - g_v(w_{i+1}, w_i)/\bar{h}_{i+1} ,$$

where we take formally: $T'_h(w_h)_{0,1} = 0$ and $T'_h(w_h)_{n,n-1} = 0$.

Then because of (1.14) $T'_h(w_h)$ is an L - matrix and:

$$(\bar{h}_{i-1}/\bar{h}_i)T'_h(w_h)_{i-1,i} + T'_h(w_h)_{i,i} + (\bar{h}_{i+1}/\bar{h}_i)T'_h(w_h)_{i+1,i} \ge c_u(x_i, w_i) ,$$

$$i=1,2,\dots,n-1,$$

hence

$$(HT'_h(w_h)H^{-1})^T e_h \ge c_* e_h .$$

Thus $T'_h(w_h)$ is an M-matrix and

$$\|((HT'_h(w_h)H^{-1})^T)^{-1}\|_\infty \le 1/c_* ,$$

i.e. (see (1.10)):

$$\|(T'_h(w_h))^{-1}\|_h \le 1/c_* ,$$

Thus T_h is a homeomorphism, [15], and because of

$$w_h - v_h = \left[\int_0^1 T'_h(v_h + s(w_h - v_h))\,ds \right]^{-1} (T_h w_h - T_h v_h) ,$$

we get (1.12). Finally, from (1.5), (1.6) we get:

$$T_h(\bar{u}e_h) \ge 0 \ge T_h(\underline{u}e_h) ,$$

hence $w_{\varepsilon,h} \in W_h$. □

Now let us consider the consistency error r_i, $i=1,2,\dots,n-1$, of the operator T_h:

$$r_i = (Tu_\varepsilon)(x_i) - T_h u_\varepsilon(x_i) .$$

We have

$$r_i = r''_i + r'_i ,$$

where

$$r''_i = \varepsilon(D''_h u_\varepsilon(x_i) - u''_\varepsilon(x_i)) ,$$

$$r'_i = f(u_\varepsilon)'(x_i) - D'_h u_\varepsilon(x_i) .$$

We shall use integral forms of r''_i and r'_i, cf. [5]. Let us introduce some notation:

$$J(i,j,k) = \int_{x_i}^{x_j} (s-x_j)^{k-1} u_\varepsilon^{(k)}(s)\,ds ,$$

$$g_{u,i} = g_u(u_\varepsilon(x_i), u_\varepsilon(x_i)) ,$$

$$g_{v,i} = g_v(u_\varepsilon(x_i), u_\varepsilon(x_i)) .$$

The following lemma is easy to prove:

Lemma 2.1 *Let (1.3) and (1.15) holds. Then for i=1,2,...,n-1 we have:*

$$r_i'' = \varepsilon(2h_i\bar{h}_i)^{-1}J(i,i-1,3) + \varepsilon(2h_{i+1}\bar{h}_i)^{-1}J(i,i+1,3) ,$$

$$r_i' = g_{u,i}\Big[(h_i - h_{i+1})(2h_{i+1}\bar{h}_i)^{-1}J(i,i+1,1) + h_{i+1}^{-1}J(i,i+1,2)\Big] +$$

$$+ g_{v,i}\Big[(h_i - h_{i+1})(2h_i\bar{h}_i)^{-1}J(i,i-1,1) + h_i^{-1}J(i,i-1,2)\Big] +$$

$$+ \bar{h}_i^{-1}\Big[\int_{u_\varepsilon(x_i)}^{u_\varepsilon(x_{i+1})} (g_{u,i} - g_u(s,u_\varepsilon(x_i)))ds +$$

$$+ \int_{u_\varepsilon(x_{i-1})}^{u_\varepsilon(x_i)} (g_{v,i} - g_v(u_\varepsilon(x_i),s))ds\Big] .$$

Let us now consider the discretization mesh I_h given by (1.8). Additionally to (1.9) assume that

(2.2) $$\lambda'(t) \le M , \quad t \in I ,$$

and that λ'' be piecewise continuous in I and

(2.3) $$|\lambda''(t)| \le M ,$$

in any subinterval of I where λ'' is continuous. From (2.2) we have

(2.4) $$M_\varepsilon h \le h_i = \lambda(t_i) - \lambda(t_{i-1}) \le Mh , \quad i=1,2,\dots,n ,$$

and from (2.3):

(2.5) $$|h_{i+1} - h_i| = |\lambda(t_{i+1}) - 2\lambda(t_i) + \lambda(t_{i-1})| \le Mh^2 ,$$

$$i=1,2,\dots,n-1 ,$$

cf. [22, Theorem 8].

Theorem 2.2 *Let* (1.3), (1,4) and (1.14-17) *hold and let the mesh* I_h *be given by* (1.8), *where* λ *satisfies* (1.9), (2.2), (2.3). *Then the convergence* (1.11) *holds.*

Proof. Since

$$r_i = T_h w_{\varepsilon,i} - T_h u_\varepsilon(x_i) , \quad i=1,2,\dots,n-1 ,$$

the result follows from Theorem 2.1, Lemma 2.1 and (2.4), (2.5). □

Finally, we shall give the theorem on the uniform convergence. The proof is obvious.

Theorem 2.3 *Let the conditions of Theorem 2.2 hold and let the following inequalities be satisfied for* $i=1,2,\dots,n-1$:

(2.6) $$\varepsilon h_i^{-1} |J(i,i-1,3)| , \ \varepsilon h_{i+1}^{-1} |J(i,i+1,3)| \le Mh^2 ,$$

(2.7) $$h_i^{-1} |J(i,i-1,1)| , \ h_{i+1}^{-1} |J(i,i+1,1)| \le M ,$$

(2.8) $$(\bar{h}_i/h_i) |J(i,i-1,2)| , \ (\bar{h}_i/h_{i+1}) |J(i,i+1,2)| \le Mh^2 .$$

Then the uniform convergence (1.13) *holds.*

The following well known schemes satisfy the conditions (1.14-17):

- the Lax - Friedrichs (LF) scheme:

$$g(u,v) = (1/2)(f(u) + f(v) + B(v-u)) ,$$

where B is such a number that

$$|b(u)| \le B , \quad u \in W .$$

– the Engquist – Oscher (EO) scheme:

$$g(u,v) = \int_0^u b_-(s)ds + \int_0^v b_+(s)ds ,$$

where

$$b_-(s) = \min\{0,b(s)\} , \quad b_+(s) = \max\{0,b(s)\} .$$

In particular, EO scheme satisfies (1.16), (1.17) since:

$$|b_-(u_1) - b_-(u_2)| \le |b(u_1) - b(u_2)| ,$$

$$|b_-(v_1) - b_-(v_2)| \le |b(v_1) - b(v_2)| ,$$

Other properties can be checked easily.

3. The indirect approach

Let $\lambda \in C^2(I)$ and let (1.8) and (1.9) hold throughout this section. Let t and $\tilde{u}$ be new variables introduced by

$$x = \lambda(t) , \quad \tilde{u}(t) = u(\lambda(t)) .$$

Let $\rho(t) = 1/\lambda'(t)$. Then the problem (1.7), (1.2) is transformed to:

$$\text{(3.1)} \qquad \tilde{T}\tilde{u} := -\varepsilon(\rho(t)\tilde{u}')' + f(\tilde{u})' + \lambda'(t)c(\lambda(t),\tilde{u}) = 0 , \quad t \in I ,$$

$$\text{(3.2)} \qquad \tilde{u}(0) = U_0 , \ \tilde{u}(1) = U_1 .$$

Here $' = d/dt$. By $\tilde{u}_\varepsilon$ we shall denote the solution to (3.1), (3.2); $\tilde{u}_\varepsilon(t) = u_\varepsilon(\lambda(t))$.

To discretize (3.1), (3.2) we use the equidistant mesh $\tilde{I}_h$. The discretization reads:

$$\text{(3.3)} \qquad \tilde{T}_h w_i := -\varepsilon\tilde{D}''_h w_i + \tilde{D}'_h w_i + \lambda'(t_i)c(\lambda(t_i),w_i) = 0 ,$$

$$i=1,2,\dots,n-1 ,$$

where

$$\tilde{D}''_h w_i = h^{-2}\Big[\rho(t_{i-1/2})(w_{i-1}-w_i) + \rho(t_{i+1/2})(w_{i+1}-w_i)\Big] ,$$

$t_{i\pm1/2} = t_i \pm h/2$, and $\tilde{D}'_h$ is the same as D'_h from Section 2 in the case of equidistant mesh. As before, w_0 and w_n in (3.3) should be replaced by U_0 and U_1, respectively.

Theorem 3.1 *Let the conditions of Theorem 2.1 hold. Then there exists a unique solution* $w_{\varepsilon,h}$ *to the problem* (3.3) *and* $w_{\varepsilon,h} \in W_h$. *Moreover, for any* w_h, v_h *the following stability inequality holds:*

$$\text{(3.4)} \qquad \sum_{i=1}^{n-1} \lambda'(t_i)|w_i - v_i| \le c_*^{-1} \|\tilde{T}_h w_h - \tilde{T}_h v_h\|_1 .$$

Proof. Let us prove (3.4). We set $z_i = w_i\lambda'(t_i)$, $y_i = v_i\lambda'(t_i)$, $i=1,2,\dots n-1$, and introduce a new operator:

$$\bar{T}_h z_i = \tilde{T}_h(z_i/\lambda'(t_i)) , \quad i=1,2,\dots,n-1 .$$

Then the Frechet derivative of $\bar{T}_h$ satisfies:

$$\|(\bar{T}'_h(z_h))^{-1}\|_1 \le 1/c_* ,$$

see the proof of Theorem 2.1. Futhermore:

$$\|z_h - y_h\|_1 \le c_*^{-1}\|\bar{T}_h z_h - \bar{T}_h y_h\|_1 ,$$

and (3.4) is immediate. The inequalities

$$\tilde{T}_h(\bar{u}e_h) \ge 0 \ge \tilde{T}_h(\underline{u}e_h)$$

complete the proof. □

Let us now consider the consistency error of the operator $\tilde{T}_h$:

$$\tilde{r}_i = (\tilde{T}\tilde{u}_\varepsilon)(t_i) - \tilde{T}_h\tilde{u}_\varepsilon(t_i) , \quad i=1,2,\dots,n-1 .$$

As before, we set:

$$\tilde{r}_i = \tilde{r}''_i + \tilde{r}'_i ,$$

where

$$\tilde{r}''_i = \varepsilon\left[\tilde{D}''_h\tilde{u}_\varepsilon(t_i) - (\rho(t)\tilde{u}'_\varepsilon(t))'_{t=t_i}\right],$$

$$\tilde{r}'_i = f(\tilde{u}_\varepsilon)'(t_i) - \tilde{D}'_h\tilde{u}_\varepsilon(t_i).$$

Additionally to (1.9) and (2.2) we shall assume that:

(3.5) $$|\lambda''(t)| \le M, \quad t \in I,$$

and that λ''' is piecewise continuous in I, in such a way that:

(3.6) $$\lambda \in C^3(t_{i-1}, t), \quad i=1,2,\dots,n,$$

(3.7) $$|\lambda'''(t)| \le M, \quad t \in (t_{i-1}, t_i), \quad i=1,2,\dots,n.$$

Let

$$\tilde{J}(i,j,k) = \int_{t_i}^{t_j} (s - t_j)^{k-1}\tilde{u}_\varepsilon^{(k)}(s)ds$$

and let $g_{u,i}$, $g_{v,i}$ have the same meaning as in Section 1. We can easily prove the following lemma, which corresponds to Lemma 2.1 in the direct approach.

Lemma 3.1 *Let* (1.3), (1.15) and (3.6) *holds. Then for* $i=1,2,\dots,n-1$ *we have*:

$$\tilde{r}''_i = \varepsilon h\Big[((\rho\tilde{u}'_\varepsilon)''(\sigma_i^+) - (\rho\tilde{u}'_\varepsilon)''(\sigma_i^-))/8 +$$

$$+ (\rho(t_{i+1/2})\tilde{u}'''_\varepsilon(\theta_i^+) - \rho(t_{i-1/2})\tilde{u}'''_\varepsilon(\theta_i^-))/48\Big],$$

where $\sigma_i^-, \theta_i^- \in (t_{i-1}, t_i)$, $\sigma_i^+, \theta_i^+ \in (t_i, t_{i+1})$,

and

$$\tilde{r}'_i = g_{u,i}\tilde{J}(i, i+1, 2)/h + g_{v,i}\tilde{J}(i, i-1, 2)/h +$$

$$+\ h^{-1}\left[\ \int_{\tilde{u}(t_i)}^{\tilde{u}_\varepsilon(t_{i+1})} (g_{u,i} - g_u(s,\tilde{u}_\varepsilon(t_i)))ds\ +\right.$$

$$\left.+\ \int_{\tilde{u}(t_{i-1})}^{\tilde{u}_\varepsilon(t_i)} (g_{v,i} - g_v(\tilde{u}_\varepsilon(t_i),s))\,ds\right].$$

Theorem 3.2 *Let* (1.3), (1.4), (1.14-17), (3.6) *and* (3.7) *hold. Then we have the convergence* (1.11).

Proof. From (3.4) it follows:

$$\sum_{i=1}^{n-1} \lambda'(t_i)\,|w_{\varepsilon,i} - \tilde{u}_\varepsilon(t_i)| \le c_*^{-1}\|\tilde{r}_h\|_1 \le M_\varepsilon\ .$$

Multiply this inequality by h, use $\tilde{u}_\varepsilon(t_i) = u_\varepsilon(x_i)$ and

$$|h\lambda'(t_i) - \bar{h}_i| \le Mh^3$$

(which is valid because of (3.6), (3.7)), and obtain

$$\|w_{\varepsilon,h} - u_{\varepsilon,h}\|_h \le M_\varepsilon h + Mh^2\ ,$$

(since $|w_{\varepsilon,i} - u_\varepsilon(x_i)| \le M$) . □

Finally we can prove:

Theorem 3.3 *Let the conditions of Theorem* 3.2 *hold and let the following inequalities be satisfied:*

(3.8) $$\varepsilon|\rho^{(k)}(t)| \le M\ ,\quad k=0,1,2\ ,$$

(3.9) $$|\tilde{u}_\varepsilon^{(k)}(t)| \le M\ ,\quad k=1,2,3\ ,$$

where $t \in (t_{i-1},t_i)$, $i=1,2,\dots,n$. *Then the uniform convergence* (1.13) *holds.*

From this theorem we can conclude how to choose λ. This can be used in the direct approach as well.

Let S denote a layer of u_ε. Very often u'_ε behaves in the following way:

$$u'_\varepsilon(x) \sim v'_\varepsilon(x) \ , \ x \in S \ ,$$

where v_ε is the corresponding layer function. If we take

$$\lambda(t) = v_\varepsilon^{-1}(t) \quad \text{for} \quad t \in \lambda^{-1}(S) \ ,$$

then (3.9) follows for k=1 and $x \in S$. With such a choice of λ we can expect that (3.9) holds for k=2,3 as well. On the other hand, outside of layers we need (2.2), (3.5), (3.7), and in these parts λ can be a suitable polynomial.

4. Two special problems

4.1 A non-turning point problem

In this section we shall consider the problem (1.1), (1.2) in the case when

$$(4.1) \qquad - b(u) \ge b_* > 0, \quad u \in W.$$

The proof of the following theorem can be found in [23], cf. [5].

Theorem 4.1. *Let* (1.3), (1.4) and (4.1) *hold. Then for* $x \in I$ *we have:*

$$(4.2) \qquad |u_\varepsilon^{(k)}(x)| \le M(1 + \varepsilon^{-k} v_\varepsilon(x)), \quad k=0,1,2,3 \ ,$$

$$v_\varepsilon(x) = \exp(-b_* x/\varepsilon) \ .$$

Thus, in this case u_ε has a boundary layer at x=0. Let us apply the direct approach. First we have to choose λ according to (4.2). In view of the discussion at the end of Section 3, we take:

$$(4.3) \qquad \lambda(t) = \begin{cases} \omega(t) \ , \ t \in [0,\alpha] \ , \\ \pi(t) := \delta(t-\alpha)^3 + \omega''(\alpha)(t-\alpha)^2/2 + \omega'(\alpha)(t-\alpha) + \omega(\alpha), \ t \in [\alpha, 1], \end{cases}$$

where $\alpha \in (0,1)$ is given, δ is determined from $\pi(1)=1$, and $\omega(t)$ corresponds to the inverse of $v_\varepsilon(x)$. Essentially the following type of function was introduced in [2]:

$$\omega(t) = \omega_1(t) := -\beta\varepsilon\ln(1 - t/\gamma) = \beta\varepsilon\ln(1 + t/(\gamma-t)),$$

where β is a positive parameter and

$$\gamma = \alpha + \varepsilon^{1/3} .$$

Similar logarithmic functions were used in [7-9]. However, it was shown in [17] that for this type of layer a certain approximation to $\omega_1(t)$ would suffice, e.g.:

$$\omega(t) = \omega_2(t) := \beta\ \varepsilon\ t/(\gamma - t) . \tag{4.4}$$

For the reasons of simplicity, we shall consider here functions of $\omega_2(t)$ - type only. Functions of this type were used in [17-25].

It is obvious that $\lambda \in C^2(I)$. From now on we shall assume that the parameter β in (4.4) is given in such a way that

$$\delta \geq 0 .$$

It is easy to see that such a positive β, independent of ε, exists. This implies:

$$0 < \lambda^{(k)}(t), \quad k=1,2, \quad t \in I .$$

Moreover, because of the special choice of γ we have

$$\lambda^{(k)}(t) \leq M, \quad k=1,2, \quad t \in I ,$$

hence the conditions (2.2), (2.3) are satisfied.

Note the following property of λ, which we shall use later

$$\exp(-b_*\lambda(t)/\varepsilon) \leq M \exp(-M/(\gamma-t)), \quad t \in (0,\alpha) .$$

Theorem 4.2 *Let* (1.3), (1,4), (4.1), (4.1), (1.14-17) *hold and let the mesh* I_h *be given by* (1.8), (4.3), (4.4). If

$$n \geq m \ln \varepsilon , \tag{4.6}$$

with an appropriate constant $m > 0$, *independent of* h *and* ε, *then we have* (1.13).

Proof. According to Theorem 2.3 we only have to prove (2.6-8). For instance, let us prove the first inequality from (2.8). The other inequalities can be proved analously. We shall use the standard technique from [2], [17-21], [23-25], which consists of the following three steps (for $i=1,2,\dots,n-1$):

$$1^\circ \quad t_{i-1} \ge \alpha - \varepsilon^{1/3} ,$$

$$2^\circ \quad t_{i-1} \le \alpha - 3h ,$$

$$3^\circ \quad \alpha - 3h < t_{i-1} < \alpha - \varepsilon^{1/3} .$$

Thus, since $h_i \le h_{i+1}$, $i=1,2,\dots n-1$, we have to prove

(4.7) $$V := (h_{i+1}/h_i) \int_{x_{i-1}}^{x_i} (s-x_{i-1}) |u''_\varepsilon(s)| ds \le Mh^2 ,$$

$$i=1,2,\dots n-1 .$$

Because of (4.2) we have

$$V \le Mh^2_{i+1}(1 + \varepsilon^{-2}) v_\varepsilon(\lambda(t_{i-1}))) .$$

We shall use this inequality in steps 1° and 2° .

In the step 1° it holds that

$$V \le Mh^2_{i+1}(1 + \varepsilon^{-2} v_\varepsilon(\lambda(\alpha - \varepsilon^{1/3})))$$

and (4.7) follows from (2.4) and (4.5).

In the step 2° we use (4.5) and

$$h_{i+1} \le Mh\varepsilon(\gamma - t_{i+1})^{-2} \le Mh\varepsilon(\gamma - t_{i-1})^{-2} ,$$

(since in this case $\alpha - t_{i+1} \ge (\alpha - t_{i-1})/3$, and (4.7) is proved again.

Finally, in the step 3° we have

$$V \le M(h^2_{i+1} + h_{i+1}\varepsilon^{-1} v_\varepsilon(x_{i-1})) \le$$

$$\le M(h^2 + h\varepsilon^{-1} v_\varepsilon(\lambda(\alpha - 3h))) \le$$

$$\le M(h^2 + h\varepsilon^{-1} \exp(-Mn)) ,$$

and (4.7) follows because of (4.6) and $\varepsilon^{1/3} < 3h$. □

Let us now consider the indirect approach. Besides (1.9), the function λ should satisfy (2.2), (3.5-7). Because of that we take λ from (4.3) with

$$(4.8) \qquad \omega(t) = \omega_3(t) := \beta\varepsilon t/(\alpha + \varepsilon^{1/4} - t),$$

where

$$\alpha = j/n \quad \text{for some} \quad j \in \{1,2,\dots n-1\} ,$$

so that (3.6) holds. Again, the positive parameter β should be chosen so that $\delta \ge 0$. The same function λ was used in [23], [24].

Theorem 4.3 *Let* (1.3), (1.4), (4.1), (1.14-17) *hold and let the mesh* I_h *be given by* (1.8), (4.3), (4.8). *Then we have* (1.13).

Proof. Because of Theorem 3.3 it is sufficient to prove (3.6) and (3.9). But these two inequalities have already been proved in [23]. □

Let us mention that a more general problem has been considered in [23] - the case when $b = b(x,u)$, and that the linear convergence uniform in ε has been proved in the equidistant norm $\|\cdot\|_h = h\|\cdot\|_1$.

4.2 A boundary shock problem

Let us now consider the case when

$$b(u) = u\, b_1(u) , \quad c(x,u) = u\, c_1(x,u)$$

$$b^* \ge b_1(u) \ge b_* > 0 , \quad u \in W ,$$

$$c^* \ge c_1(x,u) \ge c_* \ge 0 , \quad x \in I , \quad u \in W ,$$

$$U_0 = 0 , \quad U_1 > \{b^*c^* + [b^*c^*(b^*c^* - b_*c_*)]^{1/2}\}/(b^*b_*) .$$

These conditions together with (1.3) and (1.4) guarantee the following estimates of the derivatives of the exact solution:

$$|u_\varepsilon^{(k)}(x)| \le M(1 + \varepsilon^{-k}\exp(-m_0 x/\varepsilon)) , \quad x \in I , \quad k=1,2,3 ,$$

with a positive constant m_0 independent of ε. For the proof see [24], where the indirect approach was used. Here the solution behaves in the same way as in the case of the non-turning point problem, thus the same functions λ can be used, and the results analogous to Theorem 4.2 and 4.3 can be proved.

5. Concluding remarks and numerical results

Let us consider the problem from [14], [23]:

P1. $$-\varepsilon u'' - \exp(u)u' + ((\pi/2)\sin(\pi x/2))\exp(2u) = 0,$$

$$u(0) = u(1) = 0 ,$$

the solution of which is given by

$$u_\varepsilon(x) = y_\varepsilon(x) + O(\varepsilon) ,$$

where

$$y_\varepsilon(x) = -\ln[(1 + \cos(\pi x/2))\ (1 - (1/2)\exp(-x/(2\varepsilon)))] .$$

We shall compare our numerical results with $y_\varepsilon(x)$. Let

$$E_\infty = \|y_{\varepsilon,h} - w_{\varepsilon,h}\|_\infty , \quad E_h = \|y_{\varepsilon,h} - w_{\varepsilon,h}\|_h .$$

We shall use the function λ from (4.3) with (4.4) and (4.8) ih the direct and indirect approaches, respectively. In both approaches we take

$$\alpha = 0.5 , \quad \beta = 1 ,$$

getting 24 - 33 % of the mesh points in the layer, (in dependence of ε). The effect of changing α and β can be seen in [17], [23], [25], for instance. Both *EO* and *LF* schemes will be used, the latter with

$$B = 1 .$$

TABLE 5.1 P1, direct approach, EO scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_∞	8.27-2	8.49-2	8.56-2
	E_h	5.67-2	5.78-2	5.79-2
100	E_∞	4.42-2	4.48-2	4.49-2
	E_h	3.04-2	3.06-2	3.06-2

TABLE 5.2 P1, direct approach, LF scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_{∞}	8.82-2	0.104	0.105
	E_h	5.89-2	5.99-2	6.00-2
100	E_{∞}	5.41-2	5.56-2	5.60-2
	E_h	3.10-2	3.11-2	3.12-2

TABLE 5.3 P1, indirect approach, EO scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_{∞}	8.88-2	8.83-2	8.89-2
	E_h	5.99-2	6.03-2	6.03-2
100	E_{∞}	4.78-2	4.64-2	4.67-2
	E_h	3.22-2	3.19-2	3.20-2

TABLE 5.4 P1, indirect approach, LF scheme

n	ε	10^{-3}	10^{-6}	10^{-9}
50	E_{∞}	0.108	0.107	0.108
	E_h	6.23-2	6.26-2	6.26-2
100	E_{∞}	5.91-2	5.72-2	5.78-2
	E_h	3.28-2	3.25-2	3.25-2

The numerical results show more than our theory gives: the pointwise convergence uniform in ε can be observed. We can see that there is no big difference between the results of Tables 4.1-4. However, the direct approach gives somewhat better result. This confirms that the condition (4.6) of Theorem 4.2 is not essential for the proof of (1.3). It is introduced for technical reasons only - in the step 3° of the proof. (Note that the proof of Theorem 4.2 could be simpler - because of (4.6) we could use two steps only: 1° $t_{i-1} \ge \alpha - 3h$ and 2° $t_{i-1} \le \alpha - 3h$. By the proof which is given here we want to show that the condition (4.6) is needed in the step 3° only).

On the other hand, EO scheme is better than LF, which is not surprising, since in this case EO scheme reduces to the standard upwind scheme which is here more natural than LF scheme.

Similar conclusions hold in the case of the boundary shock problem, cf. [24].

Comparing the two approaches we can conclude that the proofs of the uniform convergence are simpler and easier in the case of the indirect approach. In particular, there is no need for some artifical conditions of (4.6) - type. However, the direct approach seems simpler for coding and it uses simpler functions λ. We illustarte this by the functions which should be used in the case of two boundary layers:

$$\lambda(t) = \begin{cases} \omega_2(t), & t \in [0,\alpha] \\ \pi_1(t), & t \in [\alpha, 1/2] \\ 1 - \lambda(1-t), & t \in [1/2, 1] \end{cases} \tag{5.1}$$

where $\alpha \in (0,1/2)$ and $\pi_1(t)$ is a third order polynomial, such that $\lambda \in C^2[0,1/2]$ and $\pi_1(1/2) = 1/2$. It holds that $\lambda \in C^1[0,1]$ and λ'' is discontinuous at $t=1/2$, but the direct approach allows this since (2.3) holds for $t \in I\setminus\{1/2\}$. In the indirect approach $\omega_2(t)$ should be replaced by $\omega_3(t)$, and we should use a more complicated polynomial, which connects $\omega_3(t)$ and $1 - \omega_3(1-t)$ in such a way that (3.5-7) hold.

Similar facts hold in the case of function

$$\lambda(t) = \begin{cases} \omega_2(t), & t \in [0,\alpha] \\ \pi_1(t), & t \in [\alpha, 1] \\ -\lambda(-t), & t \in [-1,0] \end{cases}$$

which should be used when the interval is $[-1,1]$ and there is an interior layer at $x=0$.

Let us use the function (5.1) to solve the following problem by the direct approach:

P2. $$-\varepsilon u'' + u u' + u + s(x) = 0, \quad u(0) = U_0, \quad u(1) = U_1,$$

where $s(x)$ and U_0, U_1 are given so that the solution reads:

$$u_\varepsilon(x) = -\exp(-x/\varepsilon) + \exp((x-1)/\varepsilon) .$$

In Table 5.5 we give the results obtained by EO scheme for α =o.25 and β=1. Let E_∞ and E_h be as before, except that here we take u_ε instead of y_ε. Note that here the error E_h decreases together with ε.

TABLE 5.5 P2, direct approach, EO scheme n=100

ε:	10^{-3}	10^{-6}	10^{-9}
E_∞	1.12-2	3.68-2	4.16-2
E_h	1.32-3	9.33-5	6.21-6

Of course, in general it is not easy to know in advance where the layers are. In the case when locations of the layers are not known, one should apply a stable equidistant scheme to locate the layers. After that, a suitable mesh generating function could be introduced.

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Rezime

METODI GENERISANJA MREŽA ZA NUMERIČKO REŠAVANJE KVAZILINEARNIH SINGULARNIH PERTURBACIONIH PROBLEMA

Kvazilinearni singularno perturbovani konturni problemi se rešavaju numerički korišćenjem diferencnih šema na specijalnim neekvidistantnim mrežama. Mreže su generisane pogodnim funkcijama koje preraspoređuju ekvidistantne tačke. Dva slična pristupa generisanju mreža su uporedena. "Uniformna" konvergencija je dokazana za dva specijalna tipa problema. Numerički rezultati ilustruju efikasnost metoda.

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