Univ. u Novom Sadu
Zb.Rad. Prirod.-Mat.Fak.
Ser.Mat. 19,2, 161-170 (1989)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

SUFFICIENT CONDITIONS FOR A UNIFORM CONVERGENCE FAMILY OF QUADRATIC SPLINE DIFFERENCE SCHEMES

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Abstract

In this paper sufficient conditions for a uniform convergence family of spline difference schemes for the problems $\varepsilon y'' + p(x)y' = f(x)$, $y(0) = \alpha$, $y(1) = \beta$, $x \in [0,1]$ are given. Numerical results are presented.

1. Derivation of schemes

We shall consider the singularly perturbed two point boundary value problems

$$(1)\qquad \begin{aligned} Ly(x) = \varepsilon y''(x) + p(x)y'(x) = f(x)\ , \quad 0 < x < 1, \\ y(0) = \alpha, \quad y(1) = \beta \end{aligned}$$

where ε is a small parameter in $(0,1]$, $p(x)$, $f(x)$ are sufficiently smooth functions and $p(x) \geq p > 0$.

AMS Mathematics Subject Classification (1980): 65L10, 65D07.
Key words and phrases: Spline collocation method, singular perturbation, fitting factor

We want to find an approximation to the solution $y(x)$ of (1) in the form of a quadratic spline $v(x) \in C^1[0,1]$ which has the form:

$$v_j(x) = v_j + (x-x_j)v_j^{(1)} + \frac{(x-x_j)^2}{2} v_j^{(2)}, \quad j = 0(1)n,$$

where $x_j = jh$, $j = 0(1)n+1$, $h = 1/(n+1)$ are the points of an equidistant mesh.

Let us define the fitting "comparison" problem associated with (1) by:

(2) $$\tilde{L}_j\tilde{y}(x) \equiv \tilde{\sigma}_j(x,\varepsilon)\tilde{y}''(x) + \tilde{p}_j(x)\tilde{y}'(x) = \tilde{f}_j(x), \quad x \in I_j, \quad j=0(1)n,$$

$$\tilde{y}(0) = \alpha, \quad \tilde{y}(1) = \beta,$$

where $\tilde{\sigma}(x,\varepsilon)$ $\tilde{p}(x)$, $\tilde{f}(x)$ are piecewisse constant approximations to a fitting factor $\sigma(x,\varepsilon)$, $p(x)$ and $f(x)$, respectively. $\tilde{\sigma}(x,\varepsilon)$ will be determined.

The unknown coefficients $v_j^{(k)}$, $k=1,2$ are determined from the following conditions.

1. $v(x)$ satisfies (2) at the points $x_{j+1/2} \equiv (x_j+x_{j+1})/2$, $j=0(1)n$ and the boundary conditions,

2. $v(x) \in C^1[0,1]$.

From the above conditions the family of difference schemes was derived in [7].

It is well-known that (1) under condition $p(x) \geq p > 0$ has a unique solution ([1], [2], [5]) $y(x)$ which displays a boundary layer at $x=0$ for small ε. For $p(x)$, $f(x) \in C^3[0,1]$ $y(x)$ can be written in the form.

$$y(x) = u(x) + w(x)$$

where

(3) $$u(x) = \varepsilon y'(0)\cdot\exp(-p(0)x/\varepsilon)/p(0),$$

(4) $$|w^{(i)}_{(x)}| \leq C\ (1 + \varepsilon^{1-i}\ \exp(-\delta x/\varepsilon)), \quad i=0(1)4,$$

where δ and C denote constants independent of ε.

Fitting factor $\tilde{\sigma}(x,\varepsilon)$ was determined in [7] from the condition that the truncation error for the boundary layer function $u(x)$ (3) is equal to zero for $p(x) = \tilde{p} =$ const. So,

$$\tilde{\sigma}_j(x,\varepsilon) = \frac{h\tilde{p}_j}{2}\tilde{w}_j, \quad x \in I_j, \quad \text{where } \tilde{w}_j = \text{cth}\,(h\tilde{p}_j/(2\varepsilon)).$$

In that way a family of implicit collocation spline difference schemes was obtained:

(5) $$\tilde{r}^-_j\, v_{j-1} + \tilde{r}^c_j\, v_j + \tilde{r}^+_j\, v_{j+1} = \tilde{q}^-_j\, \tilde{f}^-_j + \tilde{q}^c_j\, \tilde{f}^+_j\,, \quad j=1(1)n$$

$$v_0 = \alpha, \quad v_{n+1} = \beta \;;$$

where we denote by $\tilde{f}^{\pm}_j(\tilde{p}^{\pm}_j)$ an approximation $f(x)(p(x))$, $x \in I_{j\pm1}$ (for every fixed j).

$$\tilde{r}^-_j = (1-1/\tilde{w}^-_j)/h\,, \quad \tilde{r}^+_j = (1-1/\tilde{w}^+_j)/h\,, \quad \tilde{r}^c_j = -\tilde{r}^-_j - \tilde{r}^+_j\,,$$

$$\tilde{q}^-_j = 1/(\tilde{p}^-_j\tilde{w}^-_j)\,, \quad \tilde{q}^c_j = 1/(\tilde{p}^+_j\tilde{w}^+_j), \quad \tilde{w}^{\pm}_j = \operatorname{cth}\frac{h\tilde{p}^{\pm}_j}{2\varepsilon}\,.$$

Scheme (5) has an abbreviated form

(6) $$\tilde{R}v_j = \tilde{Q}\tilde{f}_j\,,$$

where

$$\tilde{R}v_j \equiv \tilde{v}^-_j\, v_{j-1} + \tilde{r}^c_j\, v_j + \tilde{r}^+_j\, v_{j+1}\,,$$

$$\tilde{Q}\tilde{f}_j \equiv \tilde{q}^-_j\, \tilde{f}^-_j + \tilde{q}^c_j\, \tilde{f}^+_j\,.$$

The choice of the approximation to $p(x)$ and $f(x)$ determines the particular scheme.

Choosing that $\tilde{p}^-_j = \tilde{p}^+_j = p_j$, $\tilde{f}^-_j = \tilde{f}^+_j = f_j$, scheme (5) becomes the Allen-Southwell-Il'in scheme for which the first order uniform convergence at the nodes has been proved in [3], [5]. The same scheme was obtained in [6] using a cubic spline $v(x) \in C^1[0,1]$.

Choosing

(7) $$\tilde{p}^{\pm}_j = p_{j\pm1/2}\,, \quad \tilde{f}^{\pm}_j = f_{j\pm1/2}$$

we get the implicit scheme which, for $\varepsilon = \tilde{\sigma}(x,\varepsilon) = 1$, becomes the scheme corresponding to the spline from [4].

The corresponding difference scheme for

(8) $$\tilde{p}^{\pm}_j = (p_{j\pm1} + p_j)/2\,, \quad \tilde{f}^{\pm}_j = (f_{j\pm1} + f_j)/2$$

has been analysed in [7]. The first order uniform and the second order classical convergence has been proved. The same order convergence (as the above schemes) has the following scheme, developed from (5) for

(9) $$\tilde{p}^{\pm}_j = p_{j\pm1}\,, \quad \tilde{f}^{\pm}_j = f_{j\pm1}\,.$$

Choosing

$$\tilde{p}_j^- = p_{j-1}, \quad \tilde{p}_j^+ = p_j, \quad \tilde{f}_j^- = f_{j-1}, \quad \tilde{f}_j^+ = f_j,$$

or

$$\tilde{p}_j^- = p_j, \quad \tilde{p}_j^+ = p_{j+1}, \quad \tilde{f}_j^- = f_j, \quad \tilde{f}_j^+ = f_{j+1},$$

the corresponding schemes have the first order uniform and classical convergence.

2. Sufficient conditions for uniform convergence

The sufficient conditions for a uniform convergence family of cubic spline difference schemes for solving (1) are given in [6].

The proof of the uniform convergence is based on the comparison method developed by Kelloyg & Tsan ([5]) and Berger et al. ([1]).

Lemma 1. *Let $\{v_j\}$ be a set of values at the grid points $\{x_j\}$, $j=0(1)n+1$ satisfying $v_0 \le 0$, $v_{n+1} \ge 0$ and $\tilde{R}v_j \ge 0$, $j=1(1)n$. Then $v_j \le 0$ for $j=0(1)n+1$.* □

Lemma 2. *If $K_1(h,\varepsilon) \ge 0$ and $K_2(h,\varepsilon) \ge 0$ are functions that satisfy:*

$$\tilde{R}(K_1(h,\varepsilon)\phi_j + K_2(h,\varepsilon)\psi_j) \ge \tilde{R}(\pm z_j) = \pm\tau_j(y)$$

where $z_j = v_j - y_j$, *then*

$$|z_j| \le K_1(h,\varepsilon)|\phi_j| + K_2(h,\varepsilon)|\psi_j| \quad \square$$

As in [1] we use two comparison functions $\phi_j = -2 + x_j$, $\psi_j = -\exp(-bx_j/\varepsilon) = -(\mu(b))^j$, where $\mu(b) = \exp(-bh/\varepsilon)$, $b > 0$ will be chosen appropriately.

Throughout the paper $\delta, M, M_1 \ldots$ will be used to denote generic constants independent of h and ε.

Lemma 3. *There are constants M_1 and M_2 such that for $h \le M_1$, $0 < b < M_2$ and $j=1(1)n$ the following holds:*

$$\tilde{R}\phi_j \ge \begin{cases} Mh/\varepsilon, & h \le \varepsilon \\ M, & \varepsilon \le h \end{cases}, \quad \tilde{R}\psi_j \ge \begin{cases} M\mu^j(b)h/\varepsilon^2, & h \le \varepsilon \\ M\mu^j(b)/h, & \varepsilon \le h \end{cases}.$$

Proof. See [7]. □

Let in (5)

(10) $|\tilde{f}_j^- - f_{j-1}| \le Mh, \quad |\tilde{f}_j^+ - f_j| \le Mh, \quad |\tilde{p}_j^- - p_{j-1}|, \quad |\tilde{p}_j^+ - p_j| \le Mh, \quad j=1(1)n,$

The truncation error for scheme (6) is:

$$\tilde{\tau}_j(y) = \tilde{R}(y_j - v_j) = \tilde{R}y_j - \tilde{Q}\tilde{f}_j =$$

$$= \tilde{R}y_j - \tilde{Q}(Ly_j) + N = \tau_j(y) + N$$

where

$$\tau_j(y) = \tilde{R}y_j - \tilde{Q}(Ly_j) = \tilde{R}y_j - \tilde{q}_j^- f_{j-1} - \tilde{q}_j^c f_j,$$

$$|N| \le Mh^2/\max(h,\varepsilon).$$

The standard Taylor expansion of $\tau_j(y)$ for fixed ε gives:

(11) $$\tau_j(y) = T_1 y_j' + T_2 y_j'' + \tilde{r}_j^+ R_2(x_j, x_{j+1}, y) + \varepsilon \tilde{\tilde{q}}_j^- R_0(x_{j-1}, x_j, y'') - p_{j-1}\tilde{\tilde{q}}_j^- R_1(x_{j-1}, x_j, y'),$$

where

(12) $$R_k(a,b,g) = g^{(k+1)}(\xi)\,\frac{(b-a)^{k+1}}{(k+1)!} = \frac{1}{k!}\int_a^b (b-s)^k g^{(k+1)}(s)ds, \quad \xi \in (a,b).$$

Lemma 4. *Let* (10) *hold, then*

$$|\tau_j(y)| \le \begin{cases} M\left(\dfrac{h^2}{\varepsilon} + \dfrac{h^2}{\varepsilon}\exp(-\delta x_j/\varepsilon)\right), & h \le \varepsilon \\ M\,(h + \exp(-\delta x_{j-1}/\varepsilon)), & \varepsilon \le h, \quad j=1(1)n. \end{cases}$$

Proof. Since $\tau_j(y) = \tau_j(u) + \tau_j(w)$ we shall estimate $\tau_j(u)$ and $\tau_j(w)$ separately.

(13) $$|T_1| = |A + (\tilde{p}_j^+ - p_j)\,\tilde{q}_j^c + \tilde{\tilde{q}}_j^-(\tilde{p}_j^- - p_{j-1})| \le \begin{cases} Mh^2/\varepsilon, & h \le \varepsilon \\ Mh, & \varepsilon \le h. \end{cases}$$

where

$$A = h(\tilde{r}_j^+ - \tilde{r}_j^-) - (\tilde{p}_j^+\tilde{q}_j^c + \tilde{p}_j^-\tilde{q}_j^-) \equiv 0, \quad j=1(1)n.$$

$$(14)\quad |T_2| = |h^2(\tilde{r}_j^+ + \tilde{r}_j^-)/2 - \varepsilon(\tilde{q}_j^- + \tilde{q}_j^c) - h\tilde{p}_{j-1}^{-}\tilde{q}_j^-)| \le \begin{cases} Mh^2/\varepsilon, & h \le \varepsilon \\ Mh\ , & \varepsilon \le h. \end{cases}$$

For the coefficients the following estimates hold:

$$(15)\quad |\tilde{r}_j^{+,c,-}| \le M/h\ ,\qquad |\tilde{q}_j^{-,c}| \le \begin{cases} Mh/\varepsilon, & h \le \varepsilon \\ M\ , & \varepsilon \le h \end{cases},\quad j=1(1)n.$$

From (13), (14), (15) and (4) we get

$$(16)\qquad |\tau_j(w)| \le \begin{cases} M\left(\dfrac{h^2}{\varepsilon} + \dfrac{h^2}{\varepsilon^2}\exp(-\delta x_j/\varepsilon)\right), & h \le \varepsilon, \\ M\,(h + \exp(-\delta x_{j-1}/\varepsilon)), & \varepsilon \le h, \quad j=1(1)n. \end{cases}$$

Let us denote by $\tau_{j,0}(u)$ the truncation error for the boundary layer function $u(x)$ (3) for $p(x) = p(0) = \text{const}$. Since $\tau_{j,0}(u) \equiv 0$, we get

$$\begin{aligned}\tau_j(u) &= \tau_j(u) - \tau_{j,0}(u) = (T_1 - T_{1,0})u_j' + (T_2 - T_{2,0})u_j'' + \\ &+ (\tilde{r}_j^- - \tilde{r}_{j,0}^-)\, R_2(x_j,x_{j-1},u) + (\tilde{r}_j^+ - \tilde{r}_{j,0}^+)\, R_2(x_j,x_{j+1},u) - \\ &- (\tilde{q}_j^- - \tilde{q}_{j,0}^-)\, \varepsilon R_0(x_j,x_{j-1},u'') - (\tilde{q}_j^- - \tilde{q}_{j,0}^-)p_{j-1}R_1(x_j,x_{j-1},u') + \\ &+ (p_{j-1} - p(0))\tilde{q}_{j,0}^-\ R_1(x_j,x_{j-1},u')\ .\end{aligned}$$

Using the estimate

$$|\tilde{\sigma}_j - \tilde{\sigma}_{j,0}| = \left|\frac{h\tilde{p}_j}{2}(\tilde{w}_j - \tilde{w}_{j,0}) + \tilde{w}_{j,0}\left(\frac{h\tilde{p}_j}{2} - \frac{hp(0)}{2}\right)\right| \le \begin{cases} \varepsilon x_j, & h \le \varepsilon \\ hx_j, & \varepsilon \le h \end{cases},$$

after some Taylor expansions, we get

$$|\tau_j(u)| \le \begin{cases} M\,\dfrac{h^2}{\varepsilon}\exp(-\delta x_j/\varepsilon), & h \le \varepsilon. \\ M\ \exp(-\delta x_{j-1}/\varepsilon), & \varepsilon \le h, \quad j=1(1)n. \end{cases}$$

The statements of Lemma 4 follow from (16) and (17). □

Theorem 1. *Let* $\{v_j\}$, $j=0(0)n+1$ *be the approximation to the exact solution* $y(x)$, $j=0(1)n+1$ *of* (1) *obtained by using* (5). *Let* (10) *hold and* $p(x), f(x) \in C^3[0,1]$, $p(x) \ge p > 0$, *then*

(18) $$|v_j - y(x_j)| \le Mh, \quad j=0(1)n+1.$$

Proof. Using the estimates from Lemma 3 and Lemma 4 by Lemma 2 we get (18).□

Theorem 2. *Let the conditions of Theorem 1 hold, then*

(19) $$|v_j - y(x_j)| \le M\frac{h^2}{\varepsilon}, \quad \text{for} \quad h \le \varepsilon, \quad j=0(1)n+1,$$

if the approximations to $p(x)$ and $f(x)$ are (7), (8) or (9).

Proof. For scheme (6) where (8) holds, (19) is proved in [7].

Since

$$(p_{j\pm1} + p_j)/2 = p_{j\pm1/2} + O(h^2), \quad \text{and}$$

$$(f_{j\pm1} + f_j)/2 = f_{j\pm1/2} + O(h^2),$$

the proof for schemes (6),(7) follows from the proof for the previous case.

Let (9) hold, then, by expanding $\tau_j(y)$ up to the derivates of the fourth order, we get

(20) $$|\tau_j(y)| \le M\frac{h^3}{\varepsilon^2} + M\frac{h^3}{\varepsilon^3}\exp(\delta x_j/\varepsilon), \quad \text{for } h \le \varepsilon.$$

Using Lemma 3 and estimate (20) we get (19).

3. Numerical examples

We shall consider the following problems.

I $\quad \varepsilon y'' + y' = x, \quad x \in [0,1],$

$\quad y(0) = y(1) = 0,$

to which the solution is

$$y(x) = (\varepsilon-1/2)(1-\exp(-x/\varepsilon))/(1-\exp(-1/\varepsilon)) - \varepsilon x + x^2/2,$$

and

II $\quad \varepsilon y'' + (1+x^2)y' = -(e^x + x^2), \quad x \in [0,1],$

$\quad y(0)=-1, \quad y(1)=0.$

For each problem the mesh length $h=1/J$ was successively halved starting wiht $J=16$ and ending wiht $J=1024$. The maximum error at all the mesh points

$$E_\infty = \max_j |y_j - v_j|$$

is listed under E_∞. The numerical rate of convergence is determined as in [2]:

$$\text{rate} = (\ln z_{k,\varepsilon} - \ln z_{k+1,\varepsilon})/\ln 2,$$

where $z_{k,\varepsilon} = \max_j |v_j^{h/2^k} - v_j^{h/2^{k+1}}|$, $k=0(1)4$,

and $v_j^{h/2^k}$ denotes the value of v_j at the mesh point x_j for the mesh length $h/2^k$.

Numerical results for scheme (8) applied to I and II are given in [7].

Tabel 1. Numerical results for (9) applied to I

	ε	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/156
K	J	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate	E_∞ rate
	16	1.5E-4	5.4E-4	1.5E-3	3.9E-3	8.3E-3	1.5E-2	2.1E-2	2.5E-2	2.7E-2
0		2.00	2.00	1.99	1.98	1.92	1.72	1.32	1.00	.95
	32	3.8E-5	1.3E-4	3.9E-4	9.8E-4	2.1E-3	4.4E-3	7.9E-3	1.1E-2	1.3E-2
1		2.00	2.00	2.00	1.99	1.97	1.92	1.72	1.34	1.03
	64	9.6E-6	3.3E-5	1.0E-4	2.4E-4	5.5E-4	1.1E-3	2.3E-3	4.0E-3	5.7E-3
2		2.00	2.00	2.00	2.00	1.99	1.98	1.91	1.71	1.34
	128	2.4E-6	8.4E-6	2.5E-5	6.2E-5	1.3E-4	2.9E-4	6.1E-4	1.1E-3	2.0E-3
3		2.00	2.00	2.00	2.00	2.00	1.99	1.98	1.92	1.72
	256	6.0E-7	2.1E-6	6.2E-6	1.5E-5	3.5E-5	7.4E-5	1.5E-4	3.1E-4	6.0E-4
4		2.00	2.00	2.00	2.00	2.00	2.00	1.99	1.98	1.92
	512	1.5E-7	5.3E-7	1.5E-6	3.8E-6	8.7E-6	1.8E-5	3.8E-5	7.8E-5	1.5E-4
	1024	3.7E-8	1.3E-7	3.9E-7	9.7E-7	2.1E-6	4.6E-6	9.7E-6	1.9E-5	3.9E-5

Table 2. Numerical results for (9) applied to II

ε / K	1/2 rate	1/4 rate	1/8 rate	1/16 rate	1/32 rate	1/64 rate	1/128 rate	1/256 rate	1/512 rate
0	2.01	2.01	2.01	1.97	1.83	1.61	1.23	1.02	.99
1	2.00	2.00	2.00	1.99	1.97	1.87	1.60	1.24	1.02
2	2.00	2.00	2.00	2.00	1.99	1.97	1.87	1.60	1.24
3	2.00	2.00	2.00	2.00	2.00	1.99	1.96	1.86	1.60
4	2.00	2.00	2.00	2.00	2.00	2.00	1.99	1.96	1.86

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Rezime

DOVOJNI USLOVI ZA UNIFORMNU KONVERGENCIJU FAMILIJE DIFERENCNIH ŠEMA IZVEDENIH POMOĆU KVADRATNOG SPLAJNA

Dati su dovoljni uslovi za uniformnu konvergenciju familije diferencnih šema za problem $\varepsilon y'' + p(x)y' = f(x)$, $y(0) = \alpha$, $y(1) = \beta$, $x \in [0,1]$. Familija je izvedena pomoću kvadratnog splajna. Prikazani su numerički rezultati.

Received by the editors June 7, 1988.