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THE NORMAL CONVERGENCE OF THE POWER SERIES IN THE n-DIMENSIONAL MIKUSINSKI DIFFERENTIATION OPERATOR

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ABSTRACT

Let $s^{(\alpha)}=s_1^{\alpha_1}\ldots s_n^{\alpha_n}$, $(\alpha_i\geq 0,\ i=1,\ldots,n)$ be the differentiation operator in the n-dimensional Mikusinski operational calculus and let $a_{(k)}$ be complex numbers depending on multi orders $(k)\in N_o^n$. The necessary and sufficient conditions for the normal convergence of power series

$$S = \sum_{(k)} a_{(k)} s^{(\alpha k)},\ ((\alpha k) = (\alpha_1 k_1,\ldots,\alpha_n,k_n)).$$

in the space of n-dimensional Mikusinski operators are given. It is shown that the convergence depends on the quasi-analyticity of certain Lelong-Carleman class, which contains the factor of convergence.

This completes the results of T.Boehme, J.Wloka, B.Stanković and the author ([1],[6],[9]).

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1. INTRODUCTION

Our terminology and notation for the n-dimensional Mikusinski operators will be as in Gutterman's paper [3], (for n=2 as in Mikusinski's book [5]), and for quasi-analytic classes of functions of n variables as in Roumieu's paper [7].

We shall give the necessary and sufficient conditions for the normal convergence of a power series in the Mikusinski differentiation operator $s^{(\alpha)}$, $S = \sum_{(k)} a_{(k)} s^{(\alpha k)}$. S is convergent in $\mathcal{M}(R^n)$ if a special class $M_{(k)} = M_{(k)}(a_{(k)})$ is not quasi-analytic (Theorem 4.1, Theorem 4.1.). Application of Lelong's theorem yields criteria in terms of the coefficients $a_{(k)}$, which are sufficient for S to be normally convergent (Theorem 4.2.).

2. n-DIMENSIONAL MIKUSINSKI OPERATORS

Let $C_o(R^n)$ denote the convolution ring of all continuous functions defined in R^n (n-dimensional Euclidean space) with the support in R^n_o, ($R^n_o = (x_1,\dots,x_n) \in R^n$, $x_i \geq 0$, $i=1,\dots,n$). The addition is the pointwise addition of functions and the convolution of $u(x_1,\dots,x_n)$ and $v(x_1,\dots,x_n)$ is the function $w(x_1,\dots,x_n)$ defined by the integral

$$w(x_1,\dots,x_n) = \int_o^{x_1} \dots \int_o^{x_n} u(x_1-t_1,\dots,x_n-t_n)v(t_1,\dots,t_n)dt_1\dots dt_n.$$

$C_o(R^n)$ has no divisors of zero [2],[3].

The field $\mathcal{M}(R^n)$ of n-dimensional Mikusinski operators is the quotient field of $C_o(R^n)$. For an operator $a \in \mathcal{M}(R^n)$, we shall use the formal notation of a quotient (the inverse operation to convolution) $a = \frac{u}{v}$, $u,v \in C_o(R^n)$ and $v(x_1,\dots,x_n) \neq 0$. Obviously, $\frac{u}{v}$ denotes the equivalence class a. To every function $u(x_1,\dots,x_n) \in C_o(R^n)$ there corresponds an operator. Thus, the set of operators contains that of functions. Further, we shall write a function $u(x_1,\dots,x_n)$ in the form u or $\{u(x_1,\ldots,x_n)\}$. By $\{u(x_1,\ldots,x_n)\}\{v(x_1,\ldots,x_n)\}$, we denote the convolution and by $\{u(x_1,\ldots,x_n)v(x_1,\ldots,x_n)\}$ the ordinary product of two functions $u(x_1,\ldots,x_n)$ and $v(x_1,\ldots,x_n)$.

We define the operator ℓ as function $\{1\}$, $\ell=\{1\}$, and the operator ℓ_i^α by $\ell_i^\alpha = \dfrac{\{\frac{x_i^\alpha}{\Gamma(\alpha+1)}\}}{\ell}$, $\alpha>0$, $i=1,\ldots,n$.

Let c be an arbitrary constant function. We define the numerical operator $[c]$ by $[c]=\dfrac{\{c\}}{\ell}$. Accordingly, the operator $I=[1]=\dfrac{\{1\}}{\ell}$ is the unity operator.

The inverse operators of ℓ and ℓ_i^α, $a>0$, $i=1,\ldots,n$, are denoted, respectively, by s and s_i^α, and are referred to as differentiation operators.

In Lemma 1. of ([3], p.473) Gutterman proved that $\ell=\ell_1\ldots\ell_n$ and $s=s_1\ldots s_n$.

For the differentiation operators s_i and the differentiable function $u(x_1,\ldots,x_n)$, we have the formula ([3], p.472):

$$s_i\{u(x_1,\ldots,x_n)\}=\{u_{x_i}(x_1,\ldots,x_n)\}+$$
$$+\ s_i\{u(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\},$$

and in general for $|r|= r_1+\ldots+r_n$, $r_i\geq 0$, $i=1,\ldots,n$,

$$D^{(r)}u = \frac{\partial^{|r|}u}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}} = s_1^{r_1}\ldots s_n^{r_n}\{u(x_1,\ldots,x_n)\}-d$$

where

$$d=\Big(\sum_{k_1=0}^{r_1}\ \ldots\ \sum_{k_n=0}^{r_n} s_1^{k_1}\ldots s_n^{k_n}\, a^{|r|}_{(k_1,\ldots,k_n)}-a^{|r|}_{(0,\ldots,0)}\Big), \tag{2.1}$$

$$a^{|r|}_{(k_1,\ldots,k_n)}=\Big\{\frac{\partial^{|r-k|}u(x_1,\ldots,x_n)}{x_1^{r_1-k_1}\ldots x_n^{r_n-k_n}}\Big\}$$

at the point $(x_1\delta_o^{k_1},\ldots,x_n\delta_o^{k_n})$, ($\delta_i^j$ is Kronecker's delta).

In Theorem 1. of ([3], p.474) Gutterman gives the conditions for an operator to be a function. The only function represented by operator (2.1.) is equal to zero.

In $M(R^n)$ we use the convergence of a sequence in the sense of the first Mikusinski convergence.

DEFINITION 2.1. *A sequence of operators* ω_k, $k=1,2,\ldots$, *converges to the operator* ω, *if there exist a function* $g(x_1,\ldots,x_n)\neq 0$ *from* $C_o(R^n)$ *and a sequence of functions* $g_k(x_1,\ldots,x_n)\in C_o(R^n)$, $k=1,2,\ldots$, *such that*

(i) $\omega g = f, \quad f(x_1,\ldots,x_n)\in C_o(R^n)$

(ii) $\omega_k g = g_k, \quad k=1,2,\ldots$

(iii) *The sequence* $g_k(x_1,\ldots,x_n)$, $k=1,2,\ldots$, *converges uniformly on every finite n-dimensional interval* $I_T=[0,T_1]\times\ldots\times[0,T_n]$ *to* $f(x_1,\ldots,x_n)$.

For the series $\sum\limits_{(k)}\omega_{(k)}$, where $\omega_{(k)}$ are operators which depend on multi order, we shall consider the normal convergence.

DEFINITION 2.2. *A series* $\sum\limits_{(k)}\omega_{(k)}$ *of n-dimensional operators depending on multi orders converges normally to the operator* ω, *if there exist a function* $g(x_1,\ldots,x_n)\neq 0$ *from* $C_o(R^n)$ *and a sequence of functions* $g_{(k)}(x\ ,\ldots,x\ )\in C_o(R^n)$, $(k)\in N_o^n$, *such that*

(i) $\omega_{(k)}g = g_{(k)}, \quad (k)\in N_o^n$

(ii) $\sum\limits_{(k)}\max\limits_{I_T}|g_{(k)}(x_1,\ldots,x_n)|$ *converges for every finite n-dimensional interval* $I_T=[0,T_1]\times\ldots\times[0,T_n]$.

Obviously, condition (ii) implies that $\sum_{(k)} g_{(k)}$ exists and is independent of the order of summation. The sum $u(x_1,\ldots,x_n) = \sum_{(k)} g_{(k)}(x_1,\ldots,x_n)$ is in $C_o(R^n)$ and $\omega = \frac{u}{g}$.

3. QUASI-ANALYTIC CLASS

Let $M_{(p)}$ be a sequence of positive real numbers depending on multi orders $(p)\in N_o^n$ and K a regular compuct set in R^n. We always suppose $M_{(o)}=1$, $0<M_{(p)}\leq\infty$ for each (p), $M_{(p)}<\infty$ for infinitely many (p).

By $\varepsilon(K,M_{(p)})$, we mean the class of all the infinitely-differentiable functions on K, such that there are constants $\beta_f>0$, and h_f depending on f and

$$\max_{x\in K} |D^{(p)}f| \leq \beta_f h_f^{|p|} M_{(p)} \tag{3.1.}$$

for each $(p)\in N_o^n$, $(p)=(p_1,\ldots,p_n)$, $|p|=p_1+\ldots+p_n$,

$$D^{(p)}f = \frac{\partial^{|p|} f}{\partial x_1^{p_1}\ldots\partial x_n^{p_n}}.$$

$\varepsilon(K,M_{(p)})$ is a vector space under the pointwise addition of functions. If K_1 and K are compact sets in R^n and (a), (b) n-tuples of real numbers such that $(x)\in K_1$ implies $(ax+b)$ (*) $\in K$, then for f in $\in\varepsilon(K,M_{(p)})$ we have $\psi=f(ax+b)\in \in(K_1,M_{(p)})$.

For an open set $\Omega\subseteq R^n$, by $\varepsilon(\Omega,M_{(p)})$ we mean the class of all functions f, such that $f\in\varepsilon(K,M_p)$ for each compact set $K\subset\Omega$.

$\mathcal{D}(\Omega,M_{(p)})$ is the set of all $f\in\varepsilon(\Omega,M_{(p)})$ which have a compact support.

A sequence $M_{(p)}$ is said to be logarithmically convex if

(*) $(ax+b)=(a_1x_1+b_1,\ldots,a_nx_n+b_n)$

(3.2) $M^2_{(p)} \leq M_{(p-q)} M_{(p+q)}$, for each $(p), (q) \in N_o^n$.

In Theorem 4. of ([7], p.158) Roumieu proved the following:

THEOREM 3.1. *If there exist constants* A *and* H *such that*

(3.3.) $M_{(p)} M_{(q)} \leq AH^{|p+q|} M_{(p+q)}$, *for each* $(p), (q) \in N_o^n$, *the vector space* $\varepsilon(\Omega, M_{(p)})$ *is an algebra under the pointwise multiplication of functions.*

REMARK. If n=1, condition (3.3.) follows from (3.2.), but for $n \geq 2$ this is not always true ([7], p.159).

DEFINITION 3.1. *A class* $M_{(p)}$ *is said to be quasi--analytic, if* $\mathcal{D}(R^n, M_{(p)}) = \{0\}$.

For n=1, a necessary and sufficient condition for the quasi-analyticity of a class M_p is known.

THEOREM 3.2. ([8], p.375) *Let* M_p *be a logarithmically convex sequence. The class* M_p *is quasi-analytic if and only if* $f \in \varepsilon(R, M_p)$, $x_o \in R$ *and* $D^p f(x_o) = 0$ *for each* $p = 0, 1, \ldots,$ *implies* $f(x) \equiv 0$ *on* R.

In the proof, we shall use that $\varepsilon(R, M_p)$ is an algebra under the pointwise multiplication of functions. In view of the remark in case n>1, we must suppose that the sequence $M_{(p)}$ satisfies (3.3.).

Now, we shall prove a similar characterization for the quasi-analyticity of a class $M_{(p)}$, for $n \geq 1$.

THEOREM 3.3. *Let the sequence* $M_{(p)}$ *satisfy inequality (3.3.). The class* $M_{(p)}$ *is not quasi-analytic if and only if there exist a function* $\psi(x_1, \ldots, x_n) \not\equiv 0$, $\psi \in \varepsilon(R^n, M_{(p)})$ *and some point* $(a_1, \ldots, a_n) \in R^n$ *such that*

$D^{(p)}\psi(x)=0$ *for each* $(x)=(x_1,\ldots,x_n)$ *with* $x_i=a_i$ *for some* $i=1,\ldots,n,$ *and each* $(p)\in N_o^n$.

PROOF. If the class $M_{(p)}$ is not quasi-analytic, then by Definition 3.1, $\mathcal{D}(R^n,M_{(p)})$ contains a nontrivial function which evidently satisfies the hypotheses of the theorem.

Conversely, let $\psi(\bar{x})\neq 0$. We may assume $(\bar{x})>(a)$. If $g(x)=\psi(x)$ for $(x)\geq(a)$ and $g(x)=0$ otherwise, then $g(x)\in\varepsilon(R^n, M_{(p)})$.

Put $h(x)=g(x+a)g(2\bar{x}-a-x)$. By Theorem 3.1. $h(x)\in\mathcal{D}(R^n, M_{(p)})$ $h(\bar{x}-a)=\psi^2(\bar{x})\neq 0$ and $h(x)$ has a compact support. Thus $h(x)$ is a nontrivial member of $\mathcal{D}(R^n,M_{(p)})$.

There is a simple characterization due to Lelong, of the quasi-analytic classes in terms of the sequences M_p.

THEOREM 3.4. (Lelong, Theorem 1. [7], p.155.) *Let* M_p *be the rectified sequence of the sequence* $\inf_{|p|=p} M_{(p)}$ *in the sense of ([7], p.154.).*
The class $M_{(p)}$ *is not quasi-analytic if and only if*

$$\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p}<\infty$$

or

$$\sum_{p=1}^{\infty}(M_p)^{-\frac{1}{p}}<\infty.$$

4. THE CONVERGENCE OF POWER SERIES IN THE OPERATOR $s^{(\alpha)}$

The normal convergence of the power series

$$S=\sum_{(k)}a_{(k)}s^{(\alpha k)} \tag{4.1.}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)\in R_o^n$, $a_{(k)}$ are complex numbers depending on multi orders (k) N_o^n and $s^{(\alpha k)}=s_1^{\alpha_1k_1}\ldots s_n^{\alpha_nk_n}$, can be characterized in terms of the quasi analyticity of Lelong-Carleman class.

Put

$$C_{(p)}=\begin{cases}\max\limits_{([\alpha k])=(p)}|a_{(k)}| & \text{if such } a_{(k)} \text{ exists}, (p)\in N_o^n\\ 0 & \text{otherwise},\end{cases}$$

$([\alpha k])=([\alpha_1 k_1],\dots,[\alpha_n k_n])$, $[y]$ denotes the biggest integer $\le y$.

THEOREM 4.1. *If the class $C^{-1}_{(p)}$ is not quasi-analytic, the series (4.1.) is normally convergent.*

THEOREM 4.1*. *If the sequence $C^{-1}_{(p)}$ satisfies (3.3.) and S is normally convergent in $M(R^n)$, then the class $C^{-1}_{(p)}$ is not quasi-analytic.*

PROOF of Theorem 4.1. Let $I_{T^*}=[0,T_1^*]\times\dots[0,T_n^*]$ be some finite interval. Since the class $C^{-1}_{(p)}$ is not quasi-analytic by Corollary 1. [7], p.156, there is a nontrivial non-negative function $\phi(x)\in\mathcal{D}(R^n,C^{-1}_{(p)})$ with the support in the interior of I_T^*. If h_ϕ is a constant, such that (3.1.) holds and $f(x)=\phi(\frac{x}{2h_\phi})$ for $\frac{x}{2h_\phi}\in I_T^*$, $f(x)=0$ otherwise, then $\ell^2 f\neq 0$ and by Theorem 1. of [3], $s^{(\alpha k)}f\in C_o(R^n)$ for each $(k)\in N_o^n$ and $(\alpha)\ge(0)$.

For each compact interval I_T there is some constant C such that

$$\max_{x\in I_T}|a_{(k)}s^{(\alpha k)}\ell^2 f|\le \max_{x\in I_T^*}|a_{(k)}\ell^{([\alpha k])-(\alpha k)}\ell^2 D^{([\alpha k])}\phi x$$

$$\times\frac{1}{(2h_\phi)^{|([\alpha k])|}}\le C\frac{1}{2^{|[\alpha k]|}}$$

$(|([\alpha k])|=[\alpha_1 k_1]+\dots+[\alpha_n k_n])$. Thus $\sum\limits_{(k)} a_{(k)}s^{(\alpha k)}$ is normally convergent in $M(R^n)$.

PROOF of Theorem 4.1*. If S is normally convergent in $M(R^n)$, there is a factor of convergence $g\in C_o(R^n)$ which must be a nontrivial infinitely differentiable function such that $D^{(p)}g(x)=0$ for every $(x)=(x_1,\ldots,x_n)$ with $x_i=0$ for some $i=1,\ldots,n$ (Theorem 1. of [3] and Definition 2.2.). Let I_T be some compact interval which contains the support number of g. By condition (ii) of Definition 2.2, there is a constant β_g such that

$$(4.3.)\qquad \max_{x\in I_T}|a_{(k)}s^{(\alpha k)}g|\leq\beta_g,\quad (k)\in N_o^n.$$

The function $\psi=\ell g$ satisfies (3.1.). Thus if $(p)=([\alpha k])$ for some $(k)\in N_o^n$ we have

$$\max_{x\in I_T}|D^{(p)}\ell g| = \max_{I_T}|\ell^{(k)-(p)}s^{(k)}\ell g|\leq\gamma C_{(p)}^{-1},$$

where γ is a constant. For other $(p)\in N_o^n$, (3.1.) is obviously satisfied. Therefore, the function $\psi=\ell g\in\varepsilon(R^n,C_{(p)}^{-1})$. The function ψ and the sequence $C_{(p)}^{-1}$ satisfy the requirements of Theorem 3., which proves that the class $C_{(p)}^{-1}$ is not quasi-analytic.

From Theorem 4.1. and Lelong's Theorem 3.4, we have another simple criteria, in terms of coefficients $a_{(k)}$, for S to be normally convergent.

THEOREM 4.2. *Let $C_{(p)}$, $(p)\in N_o^n$ be given by (4.2.). If M_p is the rectified sequence of the sequence $\inf_{|p|=P} C_{(p)}^{-1}$ in the sense of ([7], p.154.), the series S is normally convergent if one of the following conditions is satisfied:*

$$a)\quad \sum_{P=1}^{\infty}\frac{M_{P-1}}{M_P},\qquad B)\quad \sum_{P=1}^{\infty}M_P^{-\frac{1}{P}}$$

REMARK. Conversely, if the sequence $C_{(p)}^{-1}$ satisfies (3.3.) and S is normally convergent, then, by Theroem 4.1. and Theorem 3.4, conditions A and B are satisfied.

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REZIME

NORMALNA KONVERGENCIJA STEPENIH REDOVA n-DIMENZIONALNOG OPERATORA DIFERENCIRANJA MIKUSINSKOG

Pitanje konvergencije operatorskog reda $\sum_{(k)} a_{(k)} s^{(\alpha k)}$ čiji članovi zavise od uredjenih n-torki, $a_{(k)}$ su kompleksni brojevi a $s^{(\alpha k)}$ je n-dimenzionalni operator diferenciranja Mikusinskog, dovedeno je u vezu sa osobinom ne kvazi-analitičnosti klase Carleman-Lelonga. Ovo je prirodna veza jer se u toj klasi u slučaju konvergencije nalazi funkcija koja predstavlja takozvani faktor konvergencije. Dati su dovoljni uslovi za konvergenciju Teorema 4.1. i Teorema 4.2, kao i potreban i dovoljan uslov za konvergenciju uz dodatno ograničenje (3.3.) Teorema 4.1.* Ovo ograničenje je prirodno u slučaju dimenzije $n>1$, jer (3.3) nije više posledica logaritamske konveksnosti niza kao u slučaju $n=1$. Data je i jedna karakterizacija ne kvazi-analitičke klase u više dimenzionalnom slučaju (Teorema 3.3.) koja je pogodna za ovu problematiku operatorskog računa.

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