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DE HAAN'S CLASS OF DISTRIBUTIONS I

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ABSTRACT

A class of distributions, named $\pi(G)$, is defined as a generalization of de Haan's Π_g class of functions [2]. The properties and the characterisation of the class $\pi(G)$ are given. The applications of the class $\pi(G)$ are left for the second part which will be treated in the next paper.

INTRODUCTION

In a quite recently published book [1], the authors named the study of the existence of the limit

$$(1)\qquad \lim_{x\to\infty} \frac{f(\lambda x)-f(x)}{g(x)} = k(\lambda), \qquad \lambda>0,$$

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with an f measurable and g positive function, as "de Haan's theory".

For a regular varying function g the class Π_g is the class of measurable functions satisfying (1) with

$$k(\lambda) \equiv k_\rho(\lambda) = c\begin{cases}\log\lambda\ , & \rho=0\\ (\lambda^\rho-1)/\rho, & \rho\neq 0\ ,\end{cases}$$

for some constant $c\neq 0$.

In his book [2] de Haan treated this class Π_g, but he elaborated in more detail a subclass of Π_g, namely, the class Π_g but with g as a slowly varying function and c=1. The class Π_g is a proper subclass of the famous Karamata class of regularly varying functions (see [3] and [5]) and has many applications (see [1] and [2]).

Our aim is to enlarge the class Π_g to distributions (generalized functions of L. Schwartz's type [4]) and to point at some applications of such an enlarged class.

1. THE CLASS Π_g OF DISTRIBUTIONS

Writing $F(y)=f(e^y)$, $G(y)=g(e^y)$ and $K(y)=k(e^y)$, $y\in\mathbb{R}$, we obtain the "additive-argument version" of (1):

$$(2)\qquad \lim_{h\to\infty}\frac{F(y+h)-F(h)}{G(h)} = K(y),\ y\in\mathbb{R}.$$

This version suits our aim better and has been frequently used in proving the properties of elements of the class Π_g .

DEFINITION 1. Let G be a positive and measurable function. A distribution $T\in\mathcal{D}'(\mathbb{R})$ belongs to the class $\pi(G)$ if and only if for every $\varphi\in\mathcal{D}(\mathbb{R})$ and every $y\in\mathbb{R}$

$$(3)\qquad \lim_{h\to\infty} \left\langle \frac{T(x+y+h)-T(x+h)}{G(h)}, \varphi(x)\right\rangle = \langle S(x,y),\varphi(x)\rangle$$

where S(.,y) is a family of distributions which is not constant in the parameter y. If in our definition S(.,y) can be constant, then we have the class $\pi_o(G)$.

First, we notice that $\langle T(x+y),\varphi(x)\rangle=(T*\check{\varphi})(y)\in C^\infty(\mathbb{R})$, where $\check{\varphi}(x)=\varphi(-x)$ and the asterisk $*$ is the sign of the convolution. Now, (3) can be written in the from:

$$(4)\qquad \lim_{h\to\infty} \frac{(T*\check{\varphi})(y+h)-(T*\check{\varphi})(h)}{G(h)} = (S(.,y)*\check{\varphi})(0).$$

If we denote by $F(y)=(T*\check{\varphi})(y)$ and by $K(y)=(S(.,y)*\check{\varphi})(0)$, relation (4) has just the form of (2). In such a way, we related the class $\pi(G)$ of distributions with the class Π_g of functions and through relation (4) we can use all the results which concern the class Π_g, in proving properties of the class $\pi(G)$.

PROPOSITION 1. If $T\in\pi(G)$, then S(.,y) has the form $S(x,y)=k_\rho(e^y)e^{\rho x}$, $\rho\in\mathbb{R}$.

<u>Proof.</u> For every $z\in\mathbb{R}$ we have:

$$\lim_{h\to\infty} \frac{G(z+h)}{G(h)} \left\langle \frac{T(x+y+(z+h))-T(x+(z+h))}{G(z+h)}, \varphi(x)\right\rangle$$

$$= \lim_{h\to\infty} \left\langle \frac{T(x+z+y+h)-T(x+z+h)}{G(h)}, \varphi(x)\right\rangle.$$

It follows from Theorem 1.9., p.16 in [2] that there exists a $\rho\in R$ such that:

$$\lim_{h\to\infty} \frac{G(z+h)}{G(h)} = e^{\rho z}, \quad z\in R. \tag{6}$$

Relation (5) can be written now in the form:

$$e^{\rho z}\langle S(x,y),\varphi(x)\rangle = S(x+z,y),\varphi(x)\rangle. \tag{7}$$

Takin care of the existence of the derivative of a distribution, from relation (7) we can derive the following one: $\rho S(x,y)=D_x S(x,y)$, where $D_x S$ is the derivative in x of the distribution $S(x,y)$ for every $y\in R$. The unique solution of this equation is $S(x,y)=e^{\rho x}C(y)$, where $C(y)$ is a family of constant distributions. From Theorem 1.9., p.16 in [2] and relation (4), it follows that $\langle S(x,y),\varphi(x)\rangle = k_\rho(e^y)$. Hence, $C(y)=c_1 k_\rho(e^y)$ and $S(x,y)$ has the form $k_\rho(e^y)e^{\rho x}$.

REMARKS. 1) We can always suppose that G is a continuous function. Namely, if $T\in\pi(G)$, then for a $\varphi_o\in\mathcal{D}$

$$\lim_{h\to\infty}\langle \frac{T(x+y+h)-T(x+h)}{G(h)}, \varphi_o(x)\rangle = k_\rho(e^y)\langle e^{\rho x},\varphi_o(x)\rangle.$$

We can choose a y_o such that $k_\rho(e^{y_o})\langle e^{\rho x},\varphi_o(x)\rangle = 1$. Since $G(h)>0$, $h\in R$, there exists a h_o such that $\langle [T(x+y_o+h)-$ $-T(x+h)]\,\varphi_o(x)\rangle \equiv G_1(h)>0$, $h>h_o$. The function $G_1(h)$ can be enlarged for $h<h_o$: $G_1(h)=G_1(h_o)$, $h<h_o$. The function G_1 is continuous, positive and

$$\lim_{h\to\infty}\langle \frac{T(x+y+h)-T(x+h)}{G_1(h)}, \varphi(x)\rangle =$$

$$=\lim_{h\to\infty} < \frac{T(x+y+h)-T(x+h)}{G(h)}, \varphi(x) > \lim_{h\to\infty} \frac{G_1(h)}{G(h)} .$$

Since $\lim_{h\to\infty} G_1(h)/G(h)=1$, we can always replace G by G_1 in (3).

2) We have seen that G from Definition 1 satisfies relation (6). Write $G(h)=e^{\rho h}L(h)$, then there exists the following limit: $\lim_{h\to\infty} L(x+h)/L(h)=1, x\in R$, and L(lny) is a slowly verying function [5]. Then, we can always suppose that G has the form $G(h)=e^{\rho h}L(h)$.

PROPOSTITION 2. The limit given by relation (3) is uniform in y on every compact set belonging to R.

Proof. This proposition follows directly from Theorem 3. 1.16., p.139 in [1], if we use form (4) of limit (3).

PROPOSITION 3. If f is a function belonging to the class Π_g ,then $f(e^y)=F(y)$ defines a regular distribution $\tilde{F}$, which belongs to the class $\pi(G)$.

Proof. If f belongs to the class Π_g , f is measurable and $F(y)=f(e^y)$ is measurable, as well. As a consequence of Theorem 3.1.16.,p. 139 in [1], it follows that f is locally bounded ; then, the same property has F. F is locally integrable and defines a regular distribution $\tilde{F}$. Now, for a $\varphi\in\mathcal{D}$, supp $\varphi\subset[-r,r]$ and $y\in R$

$$\lim_{h\to\infty} <\frac{F(x+y+h)-F(x+h)}{G(h)}, \varphi(x)> =$$

$$= \lim_{h\to\infty} \int_{-r}^{r} \frac{F(x+y+h)-F(x+h)}{G(x+h)} \frac{G(x+h)}{G(h)} \varphi(x)\, dx.$$

We know (see [5]) that limit (6) is uniform in z on every compact set belonging to R. Taking care of Theorem 3.1.16 in [1], once again, we can apply Lebesgue's theorem to the last integral and we shall obtain for the sought limit

$$= \int_{-r}^{r} k_\rho(e^y) e^{\rho x} \varphi(x)\, dx$$

which proves our proposition.

The next example shows that we can find a locally integrable function $h: R_+ \to R$ which does not belong to any class Π_g, but the regular distribution $\tilde{h}(e^x)$ belongs to a class $\pi(G)$. Such an example is the following:

$$h(x) = x \int_{e^\alpha}^{x} g(\ln u) \frac{du}{u} + x\, g(\ln x), \; x>0,$$

g is continuous, $g(x) \le e^n, x \in J_n = (n-e^{-2n}, n+e^{-2n})$; $g(x)=0$, $x \notin J_n$, $n \in N$. Hence, $g \in L^1(-\infty,\infty)$. Now,

$$\frac{h(xt)-h(t)}{t} = x \int_{e^\alpha}^{xt} g(\ln u) \frac{du}{u} - \int_{e^\alpha}^{t} g(\ln u) \frac{du}{u} +$$

$$+ x\, g(\ln x + \ln t) - g(\ln t) \;, \; x>0$$

the two integrals have a limit when $t\to\infty$, but $x\, g(\ln x + \ln t) - g(\ln t)$ oscillates between zero and infinity. On the contrary, the function

$$h(e^x)=F(x)=e^x \int_\alpha^x g(v)\ dv + e^x g(x) = \frac{d}{dx}\left(e^x \int_\alpha^x g(v)\ dv\right)$$

defines a regular distribution belonging to $\pi(e^h)$. To show that, assume that $\varphi \in \mathcal{D}$ and supp $\varphi \subset [-r,r]$. Then, for the distribution $\tilde{F}$, limit (3) is:

$$-\lim_{h\to\infty} \int_{-r}^{r} \left[e^{x+y} \int_\alpha^{x+y+h} g(v)\ dv - e^x \int_\alpha^{x+h} g(v)\ dv\right]\varphi'(x)\ dx$$

$$= \int_\alpha^\infty g(v)\ dv\ (e^y - 1) \int_{-r}^{r} e^x \varphi(x)\ dx\ ,\ y \in R.$$

Proposition 3 makes precise in what sense the class of distributions $\pi(G)$ enlarges the class of functions Π_g. A distribution which is not regular, but belongs to a $\pi(G)$ is $T(x)=x_+^\lambda$, $\neq -1,-2,\ldots,\lambda<-1$ (see [4]). In this case, for $\varphi \in \mathcal{D}$, supp $\varphi \subset [-r,r]$, $h>r+|y|$, limit (3) is:

$$\lim_{h\to\infty} \int_{-r}^{r} \frac{(x+y+h)^\lambda - (x+h)^\lambda}{h^{\lambda-1}} \varphi(x)\ dx = \lambda\ y \int_{-r}^{r} \varphi(x)\ dx.$$

Hence, the limit distribution $S(x,y)=\lambda y$.

PROPOSITION 4. A distribution $T \in \mathcal{D}'$ belongs to the class $\pi(G)$ if and only if for every interval $I_r=[-r,r]$ there exist numerical functions F_i, $i=1,2,\ldots,m$, continuous on $[-r,\infty)$, such that for every i , $i=1,2,\ldots,m$, the limit:

$$\lim_{h\to\infty} \frac{F_i(y+h)-F_i(h)}{G(h)} = k_{i,\rho}(e^y),\ k_{i,\rho}(e^y)=c_i\ \frac{e^{\rho y}-1}{\rho}$$

is uniform for $x \in I_r$, when $h\to\infty$. The restriction of the distribution T on $[-r,\infty)$ can be given in the form

(8) $$T = \sum_{i=1}^{m} D^{j_i} F_i \ .$$

By D we denote the derivative in the sense of distributions.

Proof. First, we shall prove that the conditions are sufficient. Suppose that T is given by the sum (8), then limit (3) is:

$$\lim_{h\to\infty} \sum_{i=1}^{m} (-1)^{j_i} \int_{-r}^{r} \frac{F_i(x+y+h)-F_i(x+h)}{G(h)} \varphi^{(j_i)}(x)\ dx =$$

$$= \sum_{i=1}^{m} (-1)^{j_i} \int_{-r}^{r} k_{i,\rho}(e^y) e^{\rho x} \varphi^{(j_i)}(x)\ dx$$

$$= \int_{-r}^{r} k_\rho(e^y)\ e^{\rho x}\varphi(x)dx.$$

It follows that $T \in \pi(G)$.

Suppose, now, that $T\in\pi(G)$. Then, the set of distributions $H\equiv\{T(x+y+h)-T(x+h)/G(h), h\in[r,\infty), y\in[-r_1,r_1]\}$ is bounded in $\mathcal{D}'$. From a part of the proof of Theorem XXII, p. 51 in [4] or from a lemma proved in [6], p. 130 follows

LEMMA 1. If $t\in\pi(G)$, then for an interval $I_r\equiv[-r,r]$ and a Ω, which is a relatively compact open neighbourhood of zero in $\mathbb{R}$, there exists a $m>0$, such that for every $\varphi,\psi\in$ $\in\mathcal{D}_\Omega^m$ the function $(T*\varphi*\psi)(x)$ is continuous for $x\in[-r,\infty)$ and

$$\lim_{h\to\infty} \left[\frac{T(t+y+h)-T(t+h)}{G(h)} *\varphi(t)*\psi(t)\right](x) =$$

$$= k_\rho(e^y)[e^{\rho t}*\varphi(t)*\psi(t)](x)\ ;$$

this limit is uniform in $x\in I_r$, $y\in I_{r'}$, $r'\in R$.

To bring to an end the proof of Proposition 4, let us take relation (VI, 6; 23) from [4]

$$(9) \quad T=\Delta^{2k}*(\gamma E*\gamma E*T)-2\Delta^{k}*(\gamma E*\xi*T)+(\xi*\xi*T)$$

where E is a solution of the iterated Laplace equation $\Delta^k E=\delta$; $\nu,\xi\in\mathcal{D}_\Omega$. We have only to choose the number k large enough so that γE belongs to $\mathcal{D}^m_\Omega$. If we denote by $F_1=\gamma E*\gamma E*T$, $F_2=\gamma E*\xi*T$ and by $F_3=\xi*\xi*T$, all the functions F_i, $i=1,2,3$, are of the from $F_i=T*\varphi_i*\psi_i$, where $\varphi_i,\psi_i\in\mathcal{D}^m_\Omega$, $i=1,2,3$.

By properties of the convolution we have:

$$\frac{F_i(y+h)-F_i(h)}{G(h)} = [(\frac{T(x+y+h)-T(x+h)}{G(h)})*\varphi_i(x)*\psi_i(x)](0)$$

and by Lemma 1 it follows that

$$\lim_{h\to\infty} \frac{F_i(y+h)-F_i(h)}{G(h)} = k_\rho(e^y)[e^x*\varphi_i(x)*\psi_i(x)](0)$$

$$= k_{i,\rho}(e^y) \ .$$

uniformly in $y\in I_{r'}$.

Proposition 4 characterizes the class $\pi(G)$ as the class of distributions T given by the sum of functions F_i, $F_i(\ln y)\in\Pi_g$, where $g(t)=G(\ln t)$.

PROPOSITION 5. If $T\in\pi(G)$ and $U\in\mathcal{E}'$, then $T*U\in\pi_o(G)$ if, in addition, $(U*e^{\rho x})(0)\neq 0$, then $T*U\in\pi(G)$, as well.

Proof. If $T\in\pi(G)$, then $(T(x+y+h)-T(x+h))/G(h)$ converges in $\mathcal{D}'$ for every $y\in\mathbb{R}$. The convolution $T*U$, for a fixed U, is a continuous mapping : $\mathcal{D}'\to\mathcal{D}'$. By the properties of the convolution, we have:

$$\frac{(T*U)(t+y+h)-(T*U)(t+h)}{G(h)}=[\{\frac{T(x+y+h)-T(x+h)}{G(h)}\}*U(x)](t)$$

and this converges for every $y\in\mathbb{R}$ to:

$$k_\rho(e^y)(e^{\rho x}*U(x))(t)=(e^{\rho x}*U(x))(0)e^{\rho t}k_\rho(e^y)\ .$$

COROLLARY. If $T\in\pi(G)$, $\rho\neq 0$, then for every $k\in\mathbb{N}$, $D^kT\in\pi(G)$, as well. It follows from the facts that $D^kT=\delta^{(k)}*T$ and $\delta^{(k)}*e^{\rho x}\neq 0$. This property of the class $\pi(G)$ is very important for applications to differential equations and to other convolution equations.

2. SOME COMMENTS

1. One can ask the question why we started from the "additive-argument version" in defining the class $\pi(G)$? To answer this question, we have to remark that the membership to the class Π_g is a local property in the sense: If $f_1(t)=f_2(t)$, $t>t_0$ and if $f_1\in\Pi_g$, then $f_2\in\Pi_g$, as well. We have looked to find such a generalization of Π_g to keep that very natural property. The next proposition makes precise that the class $\pi(G)$ satisfies such a demand.

PROPOSITION 6. If for two distributions T_1 and T_2 we have $T_1 = T_2$ over an open interval (α,∞) and T_1 belongs to $\pi(G)$, then T_2 belongs to $\pi(G)$, as well.

Proof. For a $\varphi\in\mathcal{D}$ we have $\langle T_1(x+y+h),\varphi(x)\rangle=$ $=\langle T_1(x),\varphi(x-y-h)\rangle$. Denote by $\psi(x)=\varphi(x-y-h)\in\mathcal{D}$. If the supp $\varphi\subset[-r,r]$, then the supp $\psi\subset[-r+y+h,r+y+h]$. We can find h_o such that $-r-|y|+h>\alpha$, $h>h_o$. For such a h_o $\langle T_1(x+z+h),\varphi(x)\rangle=\langle T_2(x+z+h),\varphi(x)\rangle$ when $|z|<|y|$, $h>h_o$. Now, it is easy to prove that $T_2\in\pi(G)$.

The opposite of the asertion of Proposition 6 is not true. That follows from

PROPOSITION 7. Suppose that G is of the form $G(x)=L(x)$ (see remarks after Proposition 1) and that

$$\lim_{h\to\infty}\frac{T_1(x+h)-T_2(x+h)}{G(h)}=U \text{ in } \mathcal{D}',$$

where U is a constant distribution. If $T_1\in\pi(G)$, then $T_2\in\pi(G)$, as well.

Proof. For x,y belonging to compact sets in R we have the following relation:

$$\frac{T_2(x+y+h)-T_2(x+h)}{G(h)}=\frac{T_2(x+y+h)-T_1(x+y+h)}{G(h)}\frac{G(y+h)}{G(h)}+$$

$$+\frac{T_1(x+h)-T_2(x+h)}{G(h)}+\frac{T_1(x+y+h)-T_1(x+h)}{G(h)}.$$

The assertion of Proposition 7 follows directly from this relation.

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REZIME

DE HAAN-OVA KLASA DISTRIBUCIJA

De Haan [2] je definisao i izučavao klasu Π_g funkcija koje predstavljaju striktnu potklasu Karamatinih regularno promenljivih funkcija [3]. Posebno se bavio klasom Π_g kada je g sporo promenljiva funkcija. Ta klasa ima veliku primenu u raznim oblastima matematike i u raznim oblastima njene primene (vidi[1] i [2]).

U ovom radu definisana je klasa distribucija $\pi(G)$ na sledeći način:

DEFINICIJA 1. Neka je G pozitivna i merljiva funkcija Distribucija $T\in\mathcal{D}'$ pripada klasi $\pi(G)$ tada i samo tada ako za svako $\varphi\in\mathcal{D}$ i svako $y\in\mathbb{R}$ postoji granica definisana relacijom (3), gde je $S(\cdot,y)$ familija konstantnih distribucija, $S(.,y)$ različita od konstante dok y prolazi skupom $\mathbb{R}$.

Dokazane su razne osobine klase $\pi(G)$. Tako je pokazano da je $S(x,y)=c(e^{\rho y}-1)e^{\rho x}, \rho\in\mathbb{R}$; da svaka funkcija $f\in\Pi_g$ definiše regularnu distribuciju $\tilde{F}(y)=f(e^y)$ koja pripada klasi $\pi(G), G(y)=g(e^y)$. Lokalno integrabilna funkcija f ne mora pripadati klasi Π_g ako $\tilde{F}(y)=f(e^y)$ pripada klasi $\pi(G)$. Okarakterisana je klasa $\pi(G)$ kao skup distribucija $T = \sum_{i\leq m} D^i F_i$ gde $F_i(\ln y)\in\Pi_g, g(h)=G(\ln h)$. Konvolucija sa elementom $U\in\mathcal{E}'$ preslikava $\pi(G)$ u $\pi(G)$ ako $(U*e^{\rho x})(0)\neq 0$. To je važna osobina za primenu jer $\delta^{(k)}\in E'$, pa izvodi preslikavaju $\pi(G)$ u $\pi(G)$ ako je $\rho\neq 0$. Najzad je pokazano da pripadanje klasi $\pi(G)$ je lokalna osobina.

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