Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 19,1,51–64 (1989)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

A UNIFORMLY CONVERGENT DISCRETIZATION METHOD FOR A SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM OF THE FOURTH ORDER

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ABSTRACT

In this paper we consider problem (1) with a small parameter $\varepsilon>0$ and the basic assumption that $a(x)>0$. A numerical method of Petrov-Galerkin type is proposed and exponential splines as test-functions are used. Using the approach from [4] the linear convergence, uniform in ε, of the method is proved.

1. INTRODUCTION

We consider the singulary perturbed boundary value problem

AMS Mathematics Subject Classification (1980):65L10,
Key words and phrases:Singular perturbations, Petrov-Galerkin method, exponential fitting

(1) $\varepsilon u^{(4)}(x)-(a(x)u'(x))'=f(x), u(0)=u'(0)=u(1)=u'(1)=0$

with some small parameter $0<\varepsilon<<1$. Let us suppose $a(x)\geq\alpha>0$, the functions $a(x),f(x)$ are assumed to be smooth. Whereas in the literature a large number of results on second order problems exists (compare [1],[5] and the references cited therein) only few facts are known on fourth order equations. Having used a maximum principle argument, Shishkin [6] obtained some uniform with respect to the parameter ε convergence results for a problem which can be splitted immediately into a system of two second order equations. The difficulties connected with fourth order problems are due to the fact that in general a maximum principle does not hold. Our approach is based on ideas of O'Riordan and Stynes [4] and does not use any maximum principle neither for the solvability of the discrete problem nor in the convergence proof.

It is well-known that in the case of second order equations

(2) $-\varepsilon u''(x)+c(x)u'(x)+d(x)u(x)=f(x)$

the uniform convergence of the Il'in-scheme or the El-Mistikawy-Werle scheme at first was proved under the assumption $d(x)\equiv 0$. Analogously we hope that the more general problem

$$\varepsilon u^{(4)}-(ax)u')'+b(x)u=f(x)$$

can be handled without complete exponential fitting discretizing the trem $b(x)u$ completely analogously to $f(x)$. Our discretization method is a Petrov-Galerkin method using exponential splines as test functions.

Let us denote by $\|\cdot\|_k$ the norm in the Sobolev space $H^k(0,1)$ and by $\|\cdot\|_\infty$ the maximum norm. Let $(.,.)$ denote the scalar product in $L^2(0,1)$. set

$$U=\{v\in H^2(0,1) \text{ with } v(0)=v'(0)=v(1)=v'(1)\}.$$

A weak formulation of (1) is: Find $u\in U$ such that

(3) $$a(u,v):=\varepsilon(u'',v'')+(au',v')=(f,v), \text{ for all } v\in U.$$

The bilinear form $a(.,.)$ is continuous and coercive - therefore (3) admits a unique solution.

Setting $v=u$ in (3) we obtain

(4) $$\|u\|_1< \frac{1}{\alpha}\sqrt{3/2}\|f\|_o.$$

Thus our problem (3) is in some sense ε-uniformly stable, the imbedding $H^1\to C$ yields

(5) $$\|u\|_\infty<K\|f\|_\infty,$$

too. Here and in the following we denote by K in general different constants which do not depend on ε.
According to (4) a usual conform finite element method results in an ε-uniformly stable scheme (in the sense of (4) or (5)). But one con not expect uniform convergence because the solution of our original problem contains boundary layer functions. It holds [3]

(6) $$u(x,\varepsilon)=g(x,\varepsilon)+\varepsilon^{\frac{1}{2}}g_o(x,\varepsilon)\exp(-\sqrt{a(0)}x/\varepsilon^{\frac{1}{2}})+$$
$$+\varepsilon^{\frac{1}{2}}g_1(x,\varepsilon)\exp(-\sqrt{a(1)}(1-x)/\varepsilon^{\frac{1}{2}})$$

where the functions g can be expanded in an asymptotic power series with respect to $\varepsilon^{1/2}$, further it is possible to differentiate (6). In particular, $z(x)=g(x,0)$ solves the reduced problem

$$-(a(x)z')'=f(x),\ z(0)=z(1)=0.$$

From (6) we obtain the bounds

(7) (i) $\|u\|_\infty \leq K$, (ii) $\|u^{(1+\ell)}\|_\infty \leq K(\varepsilon^{-1/2})^\ell$ $(\ell=0,1,2,\ldots)$

(iii) $\|u\|_k \leq K\ \varepsilon^{-1/4}(\varepsilon^{-1/2})^{k-1}$ $(k=2,3,\ldots)$.

Thus, the standard error analysis for conform finite element methods using piecewise quadratic or piecewise cubic C^1-splines would result in

$$\|u-u_h\|_2 \leq \begin{cases} Kh\varepsilon^{-7/4} & \text{for quadratic } C^1\text{-splines} \\ Kh^2\varepsilon^{-9/4} & \text{for cubic } C^1\text{-splines} \end{cases}$$

Of course, for $\varepsilon \ll 1$ this estimate is practically not applicable and it is necessary to use special discretization techniques.

2. The discrete problem

Let some nonequidistant grid be given, i.e.

$$0=x_o<x_1<\ldots<x_{N-1}<x_N=1.$$

The corresponding step sizes we donote by $h_i=x_i-x_{i-1}$, the mesh width by $h=\max_{1\leq i\leq N} h_i$. We define by $\bar{a}(x)$ a piecewise constant approximation of $a(x)$ with $\bar{a}(x)=a_i:=a(x_i)$ for all $x\in(x_{i-1},x_i)$ and modify our original bilinear from due to

(8) $\bar{a}(u,v):=\varepsilon(u'',v'')+(\bar{a}u',v')$.

Let some finite-dimensional spaces of the same dimension S_h, T_h with $S_h\subset U$, $T_h\subset U$ be given. The discrete problem consists in the following: Find $u_h \in S_h$ such that

(9) $\bar{a}(u_h, v_h) = (\bar{f}, v_h)$ for all $v_h \in T_h$,

where $\bar{f}$ denotes a corresponding piecewise constant approximation to f. Now we introduce basic functions

$\psi_k (k=1,\dots,N-1)$ in $T_h \subset U$ by

(10) $\varepsilon \psi_k^{(4)} - \bar{a}\psi_k'' = 0$ on (x_{i-1}, x_i) for $i=1,2,\dots,N$

$\psi_k \in C^2[0,1]$ with supp $\psi_k = [x_{k-2}, x_{k+2}]$

$\psi_k(x_k) = 1.$

Thus, the test functions ψ_k are exponential C^2-splines and it is well known that ψ_k are uniquely determined (later we have to calculate them explicitely). Let us for a moment choose $S_h = T_h$ and set

$$u_h = \sum_{k=1}^{N-1} v_k \psi_k .$$

Then the unique solvability of the continuous problem results in the unique solvability of the discrete problem, therefore the corresponding system of linear equations for the unknowns v_k admits a unique solution. First we are intersted in computing approximations in the gridpoints. In principle one could use

(11) $u_h(x_k) = u_k = \psi_k(x_{k-1})v_{k-1} + v_k + \psi_k(x_{k+1})v_{k+1}$

to calculate these approximations. But it is easier to derive a linear system of equations for u_k itselves.

According to the definition of our discrete problem the function $u_h \in S_h \subset U$ (S_h arbitrary) satisfies

$$(12)\qquad (-\varepsilon\psi_k'' + \bar{a}\psi_k', u_h') = (f, \psi_k).$$

Taking into account the properties of our test functions it holds

$$(13)\qquad -\varepsilon\psi_k + \bar{a}\psi_k' = c_i^k \text{ on } [x_{i-1}, x_i]$$

$$\text{with } c_i = 0 \text{ for } i < k-2,\ i > k+3.$$

Hence, (12) results in

$$(14)\qquad \sum_{i=k-1}^{i=k+2} c_i^k (u_i - u_{i-1}) = d_k^* = (\bar{f}, \psi_k)\quad (k=1,\dots,N-1)$$

(with $c_{N+1}^{N-1} = 0$). Whereas the system for v_k represents a seven band matrix our system (14) for u_k is pentadiagonal. The system (14) does not depend on the special choice of the space S_h, thus for $S_h = T_h$ we also obtain (14) and so this system admits a unique solution, too.

Now let us set

$$t = (x - x_{i-1})/h_i,\ \lambda_i^2 = a_i/\varepsilon,\ \mu_i = h_i\lambda_i.$$

Then it is convinient to represent $\psi_k(x)$ on the interval $[x_{i-1}, x_i]$ in the form [2]

$$(15)\qquad s_i^k(x) = y_{i-1}^k(1-t) + y_i^k t + \frac{M_{i-1}^k}{\lambda_i^2}\left(\frac{\sinh\mu_i(1-t)}{\sinh\mu_i} - (1-t)\right) + \frac{M_i^k}{\lambda^2}\left(\frac{\sinh\mu_i t}{\sinh\mu_i} - t\right),$$

and ψ_k belongs to C^2 iff

$$(16)\quad \rho_i M^k_{i-1}+(\sigma_i+\sigma_{i+1})M^k_i+\rho_{i+1}M^k_{i+1}=\tau^k_{i+1}-\tau^k_i \quad (i=1,\dots,N-1)$$

with $\tau^k_i=(y^k_i-y^k_{i-1})/h_i$

$$\rho_i = \frac{\sinh\mu_i-\mu_i}{\mu_i^2\sinh\mu_i}h_i,\quad \sigma_i = \frac{\mu_i\cosh\mu_i-\sinh\mu_i}{\mu_i^2\sinh\mu_i}.$$

The differentiation results in

$$(17)\qquad c^k_i = a_i\frac{y^k_i-y^k_{i-1}}{h_i}+\varepsilon\frac{M^k_{i-1}-M^k_i}{h_i}.$$

Unfortunately, it is not so easy to calculate y^k_i, M^k_i explicitely. In the case $2<k<N-2$ it holds $M^k_\ell=0$ for $\ell<k-2$ and $\ell>k+2$, $y^k_\ell=0$ for $\ell<k-2$, $\ell>k+2$ and $y_k=1$, the five parameters M^k_{k-1}, M^k_k, M^k_{k+1}, y^k_{k-1}, y^k_{k+1} satisfy (16) for $i = k-2, k-1, k, k+1, k+2$. A tedious but simple computation yields: y^k_{k-1}, y^k_{k+1} are the solutions of

$$(18)\quad \frac{1}{\rho_{k+1}}\left[\frac{1}{h_{k+1}}+\frac{1}{h_{k+2}}+(\sigma_{k+1}+\sigma_{k+2})\frac{1}{\rho_{k+2}h_{k+2}}\right]y^k_{k+1}-\frac{1}{\rho_k}\left[\frac{1}{h_k}+\frac{1}{h_{k-1}}+(\sigma_{k-1}+\sigma_k)\frac{1}{\rho_{k-1}h_{k-1}}\right]y^k_{k-1}=\frac{1}{\rho_{k+1}h_{k+1}}-\frac{1}{\rho_k h_k}$$

$$p^*_k y^k_{k+1}+q^*_k y^k_{k-1}=2\left(\frac{1}{h_k}+\frac{1}{h_{k+1}}\right)+(\sigma_k+\sigma_{k+1})\left(\frac{1}{\rho_k h_k}+\frac{1}{\rho_{k+1}h_{k+1}}\right)$$

with

$$p_k^* = \frac{2}{h_{k+1}} - 2\,\frac{\rho_k}{\rho_{k+2}}\,\frac{1}{h_{k+2}} + \frac{\sigma_k+\sigma_{k+1}}{\rho_{k+1}}\left[\frac{1}{h_{k+2}} + \frac{1}{h_{k+2}} + (\sigma_{k+1}+\sigma_{k+2})\frac{1}{\rho_{k+2}h_{k+2}}\right]$$

$$q_k^* = \frac{2}{h_k} - 2\,\frac{k}{\rho_{k-1}}\,\frac{1}{h_{k-1}} + \frac{\sigma_k+\sigma_{k+1}}{\rho_k}\left[\frac{1}{h_k} + \frac{1}{h_{k-1}} + (\sigma_{k-1}+\sigma_k)\,\frac{1}{\rho_{k-1}h_{k-1}}\right].$$

If y_{k-1}^k, y_{k+1}^k are known we obtain M_ℓ^k by the formulas

(19) (i) $M_{k-1}^k = \dfrac{1}{\rho_{k-1}h_{k-1}}\, y_{k-1}^k$, $M_{k+1}^k = \dfrac{1}{\rho_{k+2}h_{k+2}}\, y_{k+1}^k$

(ii) $M_k^k = \dfrac{1}{\rho_k h_k} - \dfrac{1}{\rho_k}\left[\dfrac{1}{h_k} + \dfrac{1}{h_{k-1}} + (\sigma_{k-1}+\sigma_k)\,\dfrac{1}{\rho_{k-1}h_{k-1}}\right] y_{k-1}^k.$

For an equidistant grid and a(x)=a=const. the ρ_i and σ_i are constant and we get

(20) (i) $y_{k-1}^k = y_{k+1}^k = \rho/(2\sigma)$

(ii) $M_{k-1}^k = M_{k+1}^k = 1/(2h\sigma)$, $M_k^k = -1/(h\sigma)$.

It is easy to see that for the order in h it holds

$$y_{k-1}^k,\ y_{k+1}^k = O(1),\ M_\ell^k = O(h^{-2}) \quad (\ell = k-1, k, k+1)$$

corresponding to the fact that the term $a_i(y_i^k - y_{i-1}^k)/h_i$ in (17) generates the discretization of $-(au')'$ and the trem $(M_{i-1}^k - M_i^k)/h_i$ generates the discretization of $\varepsilon u^{(4)}$ (the right-hand side of our discrete problem (14) admits the order O(h)).

The basic functions $\psi_1(x)$ and $\psi_{N-1}(x)$ contain the unknown parameters $M_o^1, M_1^1, M_2^1, y_2^1$ (respectively M_{N-2}^{N-1}, M_{N-1}^{N-1}, M_N^{N-1}, y_{N-2}^{N-1}) and we have to take into consideration the boundary conditions. We obtain

$$(21)\quad \rho_1 M_o^1 - \{\frac{1}{h_2} - \frac{\rho_2}{\rho_3}\frac{1}{h_3} + \frac{\sigma_1+\sigma_2}{\rho_2}[\frac{1}{h_2} + \frac{1}{h_3} + \frac{\sigma_2+\sigma_3}{\rho_3 h_3}]\} y_2^1 = -(\frac{1}{h_1} + \frac{1}{h_2}) -$$

$$-\frac{\sigma_1+\sigma_2}{\rho_2}\frac{1}{h_2}\sigma_1 M_o^1 - \frac{\rho_1}{\rho_2}[\frac{1}{h_2} + \frac{1}{h_3} + \frac{\sigma_2+\sigma_3}{\rho_3 h_3}] y_2^1 = \frac{1}{h_1} - \frac{\rho_1}{\rho_2}\frac{1}{h_2}.$$

If M_o^1, y_2^1 are known we get

$$(22)\quad M_2^1 = \frac{1}{\rho_3 h_3} y_2^1 \; ; \; M_1^1 = \frac{1}{\rho_2 h_2} - \frac{1}{\rho_2}[-\frac{1}{h_2} + \frac{1}{h_3} + \frac{\sigma_2+\sigma_3}{\rho_3 h_3}] y_2^1.$$

Substituting $M_k^1 := M_{N-k}^{N-1}$ $(k=0,1,2)$, $y_2^1 := y_{N-2}^{N-1}$, $h_{k+1} := h_{N-k}, \rho_{k+1} := \rho_{N-k}$, $\rho_{k+1} := \sigma_{N-k}$ the corresponding system for the basic function $\psi_{N-1}(x)$ is generated. For an equidistant grid with a(x) a it holds

(23) (i) $y_2^1 = \rho\sigma/(2\sigma^2-\rho^2)$

(ii) $M_o^1 = +2(\sigma+\rho)/(h(-\rho^2+2\sigma^2))$, $M_1^1 = -(2\sigma+\rho)/(h(2\sigma^2-\rho^2))$,

$M_1^2 = \sigma/(h(2\sigma^2-\rho^2))$

Now, let us set

$$(24)\qquad d_k^* = \sum_{i=k-1}^{i=k+2} d_i^k f_i$$

for the right-hand side of our discrete problem (14). We have

$$d_i^k = \int_{x_{i-1}}^{x_i} s_i^k(x)dx = h_i \int_0^1 s_i(t)dt$$

and on such a way we get

(25) $$d_i^k = h_i \frac{y_i^k + y_{i-1}^k}{2} + \frac{M_i^k + M_{i-1}^k}{\lambda_i^2} \left(\frac{\cosh\mu_i - 1}{\mu_i \sinh\mu_i} - \frac{1}{2}\right).$$

Thus, all coefficients of our discrete problem (14) can be computed using (17),(24),(25) and the formulas (18),(19), (21), (22) for the $y_\ell^k, M_\ell^k (\ell = k-1, k, k+1)$.

3. The uniform convergence of the method

Let us define the selfadjoint operator L on the function space U by

$$(Lu,v) = (u,Lv) := \bar{a}(u,v).$$

The solution of the problem

$$L\,G_j = \delta(x - x_j),$$

a Green function, is characterized by

(26) $$\varepsilon(G_j'', w'') + (\bar{a}G_j', w') = w(x_j) \text{ for all } x \in U.$$

It holds $G_j \in H^3$ and $G_j \in C^2$. Choosing for w some finite funcions we obtain

(27) $$\varepsilon G_j^{(4)} - \bar{a}G_j'' = 0 \text{ on every subinterval } (x_{i-1}, x_i).$$

Combining (26) and (27) we get that G_j satisfies the following jump condition

$$(28)\quad S_iG_j := \lim_{x\to x_i-0}(\varepsilon G_j''' - \bar{a}G_j') - \lim_{x\to x_i+0}(\varepsilon G_j''' - \delta\bar{a}G_j') = -\delta_{ij}.$$

Thus, G_j is equivalently characterized by $G_j \in C^2[0,1]$,

$G_j(0)=G_j'(0)=G_j(1)G_j'(1)=0$ and (27),(28).

Lemma 1: G_j *belongs to the test space* T_h.

<u>Proof</u>: All functions in T_h can be represented as

$$H_h = \sum_{k=1}^{N-1} \alpha_k \psi_k.$$

New, we have to prove only that it is possible to fulfil the jump condition (28) choosing the parameters α_k in an adequate way. Let us choose $\varphi \in T_h$. From (10) it follows

$$\sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\varepsilon\psi_k^{(4)} - \bar{a}\psi_k'')\varphi \, dx = 0.$$

Integration by parts results in

$$\sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} (\sigma\psi_k''' - \bar{a}_k')\varphi \Big|_{x_{i-1}}^{x_i} - \sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\varepsilon\psi_k''' - \bar{a}\psi_k)\varphi' \, dx = 0,$$

respectively

$$(29)\quad \sum_{k=1}^{N-1} \alpha_k \sum_{i=1}^{N} (S_i\psi_k)\varphi(x_i) + \sum_{k=1}^{N} \alpha_k [(\varepsilon\psi_k'', \varphi'') + (\bar{a}\,\psi_k', \varphi')] = 0.$$

Let $\varphi_m (m=1,..,N-1)$ be some basic function of T_h with

$$\varphi_m(x_\ell) = \delta_{\ell m}.$$

Choosing $\varphi = \varphi_m$ in (29) we get

(30) $S_m H_j + \sum_{k=1}^{N-1} \alpha_k[(\varepsilon\psi_k'',\varphi_m'')+(\bar{a}\psi_k',\varphi_m')]=0.$

Now, one can choose α_k in such a way that $S_m H_j=-\delta_{mj}$ because (30) represents a linear system for the unknowns α_k which is equivalent to the system for the unknowns v_k and therefore uniquely solvable. □

Lemma 2: *For the error at the grid points it holds*

(31) $u(x_i)-u_i=(u',(\bar{a}-a)G_i')+(f-\bar{f},G_i),$

Proof: Let us start with

$$u(x_i)-u_i((u-u_h)(x),\delta(x-x_i))=((u-u_h)(x),LG_i)=\bar{a}(u-u_h,G_i).$$

A splitting yields

$$u(x_i)-u_i=\bar{a}(u,G_i)-\bar{a}(u_h,G_i)$$

$$=\bar{a}(u,G_i)-a(u,G_i)+a(u,G_i)-\bar{a}(u_h,G_i)$$

$$=((\bar{a}-a)u',G_i')+(f,G_i)-(\bar{f},G_i)$$

taking into consideration $G_i \in T_h$. □

THEOREM 1: *Let us assume*

$$\|\bar{a}-a\|_\infty<Kh,\|\bar{f}-f\|_\infty<Kh.$$

Then, the above defined method converges uniformly with respect to ε *at the gridpoints and the following error estimation holds*

(32) $\max_i |u(x_i)-u_i|<Kh,$

where K *does not depend on* ε, h.

Proof: Let us note that we have already proved $\|u\|_1 \leq K$. Analogously, from (26) it follows $\|G_i\|_1 \leq K, \|G_i\|_\infty \leq K$. Now, the statement follows directly from (31).□

In order to approximate u(x) for all x we use the linear interpolate πu_h which satisfies $(\pi u_h)(x_i)=u_i$.

Lemma 3: *For the linear interpolate it holds*

$$\|u-\pi u_h\|_\infty \leq Kh$$

that means, u is approximated uniformly with respect to ϵ *in the* L^∞ *-norm.*

Proof: Let us fix some $x \in (x_{i-1}, x_i)$. Then we have

$$u(x)-\pi u_h(x)=[u(x)-u(x_{i-1})+u(x_{i-1})-u_{i-1}](x-x_{i-1})/(x_i-x_{i-1})+$$

$$+[u(x)-u(x_i)+u(x_i)-u_i](x-x_i)/(x_{i-1}-x_i).$$

Thus, from theorem 1 and $\|u'\|_\infty \leq K$ the desired result follows.□

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REZIME

JEDAN UNIFORMNO KONVERGENTAN METOD DISKRETIZACIJE ZA SINGULARNO PERTURBOVANI KONTURNI PROBLEM ČETVRTOG REDA

U radu se posmatra problem (1) sa malim parametrom $\varepsilon>0$ i osnovnom pretpostavkom $a(x)>0$. Predložen je numerički postupak Petrova-Galjerkina koji koristi eksponencijalne splajnove kao test-funkcije. Korišćenjem pristupa iz rada [4] za postupak je dokazana linearna konvergencija, uniformna po ε.

Received by the editors June 22, 1987.