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AFFINE CONNECTIONS IN THE FINSLER SPACE

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ABSTRACT

In the Finsler space some general connection coefficients are introduced which depend on seven arbitrarily chosen parameters. Special cases of these are Cartan connection coefficients and some of the recurrent Finsler spaces. All of them have the property that the angle between two vectors by parallel displacement remains unchanged. It is shown that there are 2^7 essentially different types of connection coefficients of this kind. For the general case some fundamental relations are obtained.

§1. CONNECTION COEFFICIENTS IN D RECURRENT FINSLER SPACES

Let the metric function in the Finsler space F_N be denoted by $L(x,\dot{x})$, and the metric tensor by $g_{\alpha\beta}(x,\dot{x})$. Then,

$$(1.1)\qquad g_{\alpha\beta}(x,\dot{x}) = \dot{\partial}_\alpha\dot{\partial}_\beta F(x,\dot{x}),\qquad F(x,\dot{x}) = 2^{-1}L^2(x,\dot{x}).$$

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$L(x,\dot{x})$ is homogeneous of degree one in $\dot{x}$, $g_{\alpha\beta}(x,\dot{x})$ and any other tensor or vector field is supposed to be homogeneous of degree zero in $\dot{x}$.

If we denote by D the absolute differential which corresponds to the change of line element from $(x,\dot{x})$ to $(x+dx,\dot{x}+d\dot{x})$, then for arbitrary tensor field $T^{\alpha}_{\beta}(x,\dot{x})$ we define

(1.2) $$DT^{\alpha}_{\beta}=dT^{\alpha}_{\beta}+(\Gamma^{*\alpha}_{\iota\gamma}T^{\iota}_{\beta}-\Gamma^{*\iota}_{\beta\gamma}T^{\alpha}_{\iota})dx^{\gamma}+(A^{\alpha}_{\iota\gamma}T^{\iota}_{\beta}-A^{\iota}_{\beta\gamma}T^{\alpha}_{\iota})\bar{D}\ell^{\gamma},$$

where

(1.3) $$\ell^{\gamma}=\frac{\dot{x}^{\gamma}}{L(x,\dot{x})}$$

(1.4) $$\bar{D}\ell^{\gamma}=d\ell^{\gamma}+L^{-1}N^{\gamma}_{\beta}dx^{\beta}+M^{\gamma}_{\beta}\bar{D}^{\beta}.$$

N^{γ}_{β} is the coefficient of non-linear connection homogeneous of degree one in $\dot{x}$ and M^{γ}_{β} is a tensor. With respect to the coordinate transformation

(1.5) $$x^{\alpha'}=x^{\alpha'}(x^{1},x^{2},\dots,x^{N})\Leftrightarrow x^{\alpha}=x^{\alpha}(x^{1'},x^{2'},\dots,x^{N'})$$

$$\alpha'=1',2',\dots,N' \qquad \alpha=1,2,\dots,N,$$

where both families of functions are sufficiently times differentiable and where

(1.6) $$\dot{x}^{\alpha'}=\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\dot{x}^{\alpha} \quad\Leftrightarrow\quad \dot{x}^{\alpha}=\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\dot{x}^{\alpha'}$$

(1.7) $$\frac{\partial}{\partial\dot{x}^{\theta'}}=\frac{\partial x^{\theta}}{\partial x^{\theta'}}\frac{\partial}{\partial\dot{x}^{\theta}}\ ,\ \frac{\partial}{\partial x^{\theta'}}=\frac{\partial x^{\theta}}{\partial x^{\theta'}}\frac{\partial}{\partial x^{\theta}}+\frac{\partial^{2}x^{\theta}}{\partial x^{\theta'}\partial x^{\alpha'}}\dot{x}^{\alpha'}\frac{\partial}{\partial\dot{x}^{\theta}},$$

we have

(1.8) $$N^{\alpha'}_{\gamma'}=\frac{\partial^{2}x^{\alpha}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\dot{x}^{\beta'}+N^{\alpha}_{\gamma}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}},$$

(1.9) $$M^{\alpha'}_{\gamma'}=M^{\alpha}_{\gamma}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}.$$

From (1.2) we have

$$(1.10)\qquad Dg_{\alpha\beta} = dg_{\alpha\beta} - (\overset{*}{\Gamma}{}^{\delta}_{\alpha\gamma}\, g_{\delta\beta} + \overset{*}{\Gamma}{}^{\delta}_{\beta\gamma} g_{\alpha\delta})dx^{\gamma} - (A^{\delta}_{\alpha\gamma} g_{\delta\beta} + A^{\delta}_{\beta\gamma} g_{\alpha\delta})\bar{D}\ell^{\gamma}.$$

Using (1.4) and (1.10) we get

$$(1.11)\qquad Dg_{\alpha\beta} = g_{\alpha\beta|\gamma} dx^{\gamma} + g_{\alpha\beta}\big|_{\gamma} \bar{D}\ell^{\gamma},$$

where

$$(1.12)\qquad g_{\alpha\beta|\gamma} = \partial_{\gamma} g_{\alpha\beta} - \dot{\partial}_{\delta} g_{\alpha\beta} N^{\delta}_{\gamma} - \Gamma^{*}_{\alpha\beta\gamma} - \Gamma^{*}_{\beta\alpha\gamma}$$

$$(1.13)\qquad g_{\alpha\beta}\big|_{\gamma} = L(\dot{\partial}_{\delta} g_{\alpha\beta})(\delta^{\delta}_{\gamma} - M^{\delta}_{\gamma}) - A_{\alpha\beta\gamma} - A_{\beta\alpha\gamma}.$$

Definition 1.1. The Finsler space is D recurrent if there exist vector fields $\lambda_{\gamma} = \lambda_{\gamma}(x,\dot{x})$ and $\mu_{\gamma} = \mu_{\gamma}(x,\dot{x})$ such that

$$(1.14)\qquad Dg_{\alpha\beta} = K(x,\dot{x},dx,\bar{D}\ell) g_{\alpha\beta}\ ,$$

where

$$(1.15)\qquad K(x,\dot{x},dx,\bar{D}\ell) = \lambda_{\gamma} dx^{\gamma} + \mu^{\gamma} \bar{D}\ell^{\gamma}\ .$$

Such a Finsler space we shall denote by $F_N(D)$.

Theorem 1.1. In the space $F_N(D)$ in which $\lambda_{\gamma}, \mu_{\gamma}, N^{\gamma}_{\beta}, M^{\gamma}_{\beta}$ are given and they satisfy (1.4) and (1.10)-(1.15) the connection coefficients $\Gamma^{*\beta}_{\alpha\gamma}$, $A^{\beta}_{\alpha\gamma}$ depend on arbitrarily chosen tensors $\theta_{\alpha\beta}$, $\tilde{\Gamma}^{\beta}_{\alpha\gamma}$ and $\tilde{A}^{\beta}_{\alpha\gamma}$, where

$$(1.16)\qquad \begin{aligned} &(a)\quad \theta_{\alpha\beta} = \theta_{\beta\alpha} \\ &(b)\quad \tilde{\Gamma}^{\beta}_{\alpha\gamma} = \Gamma^{*\beta}_{\alpha\gamma} - \Gamma^{*\beta}_{\gamma\alpha} \\ &(c)\quad \tilde{A}^{\beta}_{\alpha\gamma} = A^{\beta}_{\alpha\gamma} - A^{\beta}_{\gamma\alpha} \end{aligned}$$

and have the form

$$\Gamma^*_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma} - 2^{-1}(N^{\delta}_{\gamma}\dot{\partial}_{\delta}g_{\alpha\beta} + N^{\delta}_{\alpha}\dot{\partial}_{\delta}g_{\beta\gamma} - N^{\delta}_{\beta}\dot{\partial}_{\delta}g_{\gamma\alpha})$$

$$(1.17)\qquad -2^{-1}[\lambda_{\gamma}(g_{\alpha\beta}+\theta_{\alpha\beta})+\lambda_{\alpha}(g_{\beta\gamma}+\theta_{\beta\gamma})-\lambda_{\beta}(g_{\gamma\alpha}+\theta_{\gamma\alpha})] + 2^{-1}(\tilde{\Gamma}_{\alpha\beta\gamma}-\tilde{\Gamma}_{\beta\gamma\alpha}+\tilde{\Gamma}_{\gamma\alpha\beta})$$

$$-[\theta_{\alpha\beta}(\Gamma^{*o}_{o\gamma}-L^{-1}N^{o}_{\gamma})+\theta_{\beta\gamma}(\Gamma^{*o}_{o\alpha}-L^{-1}N^{o}_{\alpha})-\theta_{\gamma\alpha}(\Gamma^{*o}_{o\beta}-L^{-1}N^{o}_{\beta})]$$

($\gamma_{\alpha\beta\gamma}$ is the Christoffel symbol),

$$A_{\alpha\beta\gamma} = 2^{-1}L\ [(\delta^{\delta}_{\gamma}-M^{\delta}_{\gamma})\dot{\partial}_{\delta}g_{\alpha\beta}+(\delta^{\delta}_{\alpha}-M^{\delta}_{\alpha})\dot{\partial}_{\delta}g_{\beta\gamma})-(\delta^{\delta}_{\beta}-M^{\delta}_{\beta})\dot{\partial}_{\delta}g_{\gamma\alpha}]$$

$$-2^{-1}(\mu_{\gamma}g_{\alpha\beta}+\mu_{\alpha}g_{\beta\gamma}-\mu_{\beta}g_{\alpha\gamma}) + 2^{-1}(\tilde{A}_{\alpha\beta\gamma}-\tilde{A}_{\beta\gamma\alpha}+\tilde{A}_{\gamma\alpha\beta})$$

$$(1.18)\qquad -2^{-1}[(\mu_{\gamma}+2\ell_{\gamma})\theta_{\alpha\beta}+(\mu_{\alpha}+2\ell_{\alpha})\theta_{\beta\gamma}-(\mu_{\beta}+2\ell_{\beta})\theta_{\alpha\gamma}]$$

$$-[\theta_{\alpha\beta}(A^{o}_{o\gamma}-M^{o}_{\gamma})+\theta_{\beta\gamma}(A^{o}_{o\alpha}-M^{o}_{\alpha})-\theta_{\gamma\alpha}(A^{o}_{o\beta}-M^{o}_{\beta})].$$

Proof. From the relation $g_{\alpha\beta}\ell^{\alpha}\ell^{\beta} = 1$, using (1.14) and

$$(1.19)\qquad D\ell^{\alpha} = d\ell^{\gamma}+\Gamma^{*\gamma}_{o\beta}dx^{\beta}+A^{\gamma}_{o\beta}\bar{D}\ell^{\gamma}$$

("o" means the contraction by ℓ) we obtain

$$(1.20)\qquad [\lambda_{\gamma}+2\ell_{\kappa}(\Gamma^{*\kappa}_{o\gamma}-L^{-1}N^{\kappa}_{\gamma})]dx^{\gamma}+[\mu_{\gamma}+2\ell_{\gamma}+2\ell_{\kappa}(A^{\kappa}_{o\gamma}-M^{\kappa}_{\gamma})]\bar{D}\ell^{\gamma} = 0.$$

This is the crucial relation which shows that dx^{γ} and $\bar{D}\ell^{\gamma}$ are not independent. For $\lambda_{\gamma}=0$, $\mu_{\gamma}=0$, $\Gamma^{*\alpha}_{o\gamma}=L^{-1}N^{\alpha}_{\gamma}$, $A^{\alpha}_{o\gamma}=M^{\alpha}_{\gamma}$ (1.20) reduces to the well known formula $\ell_{\alpha}D\ell^{\alpha}=0$ in the non-recurrent Finsler space. If we multiply (1.20) with $\theta_{\alpha\beta}(x,\dot{x})$, $(\theta_{\alpha\beta}=\theta_{\beta\alpha})$ and add to the right hand side of (1.14) using (1.11), we have

$$(1.21)\qquad g_{\alpha\beta|\gamma}dx^{\gamma}+g_{\alpha\beta}|_{\gamma}\bar{D}\ell^{\gamma}=\lambda_{\gamma}(g_{\alpha\beta}+\theta_{\alpha\beta})dx^{\gamma}+[\mu_{\gamma}g_{\alpha\beta}+(\mu_{\gamma}+2\ell_{\gamma})\theta_{\alpha\beta}]\bar{D}\ell^{\gamma}$$

$$+2\theta_{\alpha\beta}\ell_{\kappa}(\Gamma^{*\kappa}_{o\gamma}-L^{-1}N^{\kappa}_{\gamma})dx^{\gamma}+2\theta_{\alpha\beta}\ell_{\kappa}(A^{\kappa}_{o\gamma}-M^{\kappa}_{\gamma})\bar{D}\ell^{\gamma}.$$

From (1.21) we get

(1.22) $$g_{\alpha\beta|\gamma}=\lambda_\gamma(g_{\alpha\beta}+\theta_{\alpha\beta})+2\theta_{\alpha\beta}\ell_\kappa(\Gamma^{*\kappa}_{0\gamma}-L^{-1}N^{\kappa}_{\gamma})$$

(1.23) $$g_{\alpha\beta}\big|_\gamma = \mu_\gamma g_{\alpha\beta}+(\mu_\gamma+2\ell_\gamma)\theta_{\alpha\beta} \;\; +2\theta_{\alpha\beta}\ell_\kappa(A^{\kappa}_{0\gamma}-M^{\kappa}_{\gamma}).$$

From (1.22), (1.12) and (1.16b) we obtain (1.17). From (1.23), (1.13) and (1.16c) we get (1.18).

Definition 1.2. The D recurrent Finsler space in which the connection coefficients are given by (1.17) and (1.18) will be denoted by $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(N,M)$.

Special cases of connection coefficients are obtained if we take $(N^\alpha_\gamma,0)$, $(L\Gamma^{*\alpha}_{0\gamma},A^\alpha_{0\gamma})$ or $(L\Gamma^{*\alpha}_{0\gamma},0)$ instead of an arbitrarily chosen pair $(N^\alpha_\gamma,M^\alpha_\gamma)$ which satisfies (1.8) and (1.9).

Lemma 1.1. With respect to the coordinate transformation (1.5) and (1.6) $\Gamma^{*\beta}_{\alpha\gamma}$ and $\Gamma^{*\beta}_{0\gamma}$ (both homogeneous of degree zero in $\dot{x}$) are transformed in the following way:

(1.24) $$\Gamma^{*\alpha'}_{\beta'\gamma'} = \frac{\partial^2 x^\alpha}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha} + \Gamma^{*\alpha}_{\beta\gamma}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}$$

(1.25) $$\Gamma^{*\alpha'}_{0\gamma'} = \frac{\partial^2 x^\alpha}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^{\beta'}}{\partial x^\beta}\ell^\beta + \Gamma^{*\alpha}_{0\gamma}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}$$

and $A_{\alpha\beta\gamma}$ is a tensor.

Proof. By direct calculation using (1.8), (1.9) and the known transformation law of $\gamma^\beta_{\alpha\gamma}$, we obtain (1.24) and (1.25). As can be seen from (1.18), $A_{\alpha\beta\gamma}$ is the sum of tensors, so it is also a tensor.

Lemma 1.2. In $F_N(D)$, we have

(1.26) $$Dg^{\alpha\beta} = -Kg^{\alpha\beta}$$

Proof. As $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$ we obtain

$$(1.27) \qquad (Dg^{\alpha\beta})g_{\beta\gamma}+g^{\alpha\beta}Dg_{\beta\gamma} = D\delta^{\alpha}_{\gamma}.$$

From

$$D\delta^{\alpha}_{\gamma} = d\delta^{\alpha}_{\gamma}+(\Gamma^{*\alpha}_{\kappa\beta}\delta^{\kappa}_{\gamma}-\Gamma^{*\kappa}_{\gamma\beta}\delta^{\alpha}_{\kappa})dx^{\beta}+(A^{\alpha}_{\kappa\beta}\delta^{\kappa}_{\gamma}-A^{\kappa}_{\gamma\beta}\delta^{\alpha}_{\kappa})\bar{D}\ell^{\beta} = 0$$

and (1.27) we obtain (1.26).

Theorem 1.2. In $F_N(D)$ the parallel displacement is affine, i.e. the angle between two vectors does not change.

Proof. Let us denote by $|\xi|$ the length of vector ξ^{α}. Then, using (1.14), we have

$$D|\xi|^2 = 2|\xi|D|\xi| = D(g_{\alpha\beta}\xi^{\alpha}\xi^{\beta}) = Kg_{\alpha\beta}\xi^{\alpha}\xi^{\beta}+2g_{\alpha\beta}(D\xi^{\alpha})\xi^{\beta}.$$

As by the parallel displacement $D\xi^{\alpha}=D\xi^{\beta}=0$, we obtain

$$2|\xi|D|\xi| = Kg_{\alpha\beta}\xi^{\alpha}\xi^{\beta} = K|\xi|^2 \Rightarrow$$

$$(1.28) \qquad D|\xi| = 2^{-1}K|\xi|.$$

If $\theta\colon \sphericalangle(\xi,\eta)$, where ξ^{α} and η^{α} are two vector fields, then

$$D(g_{\alpha\beta}\xi^{\alpha}\eta^{\beta}) = D(|\xi||\eta|\cos\theta) =$$

$$(|\eta|D|\xi|+|\xi|D|\eta|)\cos\theta+|\xi||\eta|D\cos\theta.$$

Using (1.14) and (1.28) for the parallel displacement of vectors ξ^{α} and η^{α}, we obtain

$$D(g_{\alpha\beta}\xi^{\alpha}\eta^{\beta})=Kg_{\alpha\beta}\xi^{\alpha}\eta^{\beta} = K|\xi||\eta|\cos\theta = K|\xi||\eta|\cos\theta+|\xi||\eta|D\cos\theta \Rightarrow$$

$$D\cos\theta = 0.$$

§ 2. CONNECTION COEFFICIENTS IN THE SPACE $F_N(D,\lambda,\mu,\theta,\Gamma,A)(N,M)$ WHERE $L^{-1}N^{\alpha}_{\beta} = \Gamma^{*\alpha}_{o\beta}$ AND $M^{\alpha}_{\beta} = A^{\alpha}_{o\beta}$. Since for any tensor field $T^{\alpha}_{\beta}(x,\dot{x})$ in $F_N(D)$ we have (1.2), it is most natural that $\bar{D}\ell^{\alpha}$ instead of (1.4) be defined in the following form

$$D\ell^{\alpha} = d\ell^{\alpha}+\Gamma^{*\alpha}_{o\beta}dx^{\beta}+A^{\alpha}_{o\beta}D\ell^{\beta}. \tag{2.1}$$

In this case $\Gamma^{*}_{\alpha\beta\gamma}$, $A_{\alpha\beta\gamma}$ determined by (1.17), (1.18) respectivly have another form, because here

$$\text{a)}\quad L^{-1}N^{\alpha}_{\beta} = \Gamma^{*\alpha}_{o\beta} \qquad \text{b)}\quad M^{\alpha}_{\beta} = A^{\ \alpha}_{o\beta}. \tag{2.2}$$

Lemma 2.1. Under condition (2.2a), the connection coefficients $\Gamma^{*}_{\alpha\beta\gamma}$ are functions of λ, θ and $\tilde{\Gamma}$, determined by (2.3)-(2.6). Under condition (2.2b), $A_{\alpha\beta\gamma}$ are functions of μ, θ and $\tilde{A}$, determined by (2.7)-(2.10).

Proof. From (1.17) and (2.2a) we have

$$\Gamma^{*}_{\alpha\beta\gamma} = {}^{1}\Gamma^{*}_{\alpha\beta\gamma}+{}^{2}\Gamma^{*}_{\alpha\beta\gamma}+{}^{3}\Gamma^{*}_{\alpha\beta\gamma}+{}^{4}\Gamma^{*}_{\alpha\beta\gamma}, \tag{2.3}$$

where

(2.4)

(a) $${}^{1}\Gamma^{*}_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma}-2^{-1}Q_{\alpha\beta\gamma}({}^{1}\Gamma^{*})$$

(b) $${}^{2}\Gamma^{*}_{\alpha\beta\gamma} = -2^{-1}(\lambda_{\gamma}g_{\alpha\beta}+\lambda_{\alpha}g_{\beta\gamma}-\lambda_{\beta}g_{\gamma\alpha})-2^{-1}Q_{\alpha\beta\gamma}({}^{2}\Gamma^{*})$$

(c) $${}^{3}\Gamma^{*}_{\alpha\beta\gamma} = -2^{-1}(\lambda_{\gamma}\theta_{\alpha\beta}+\lambda_{\alpha}\theta_{\beta\gamma}-\lambda_{\beta}\theta_{\gamma\alpha})-2^{-1}Q_{\alpha\beta\gamma}({}^{3}\Gamma^{*})$$

(d) $${}^{4}\Gamma^{*}_{\alpha\beta\gamma} = -2^{-1}(\tilde{\Gamma}_{\alpha\beta\gamma}-\tilde{\Gamma}_{\beta\gamma\alpha}+\tilde{\Gamma}_{\gamma\alpha\beta})-2^{-1}Q_{\alpha\beta\gamma}({}^{4}\Gamma^{*})$$

$$Q_{\alpha\beta\gamma}({}^{k}\Gamma^{*}) = L(\dot{\partial}_{\delta}g_{\alpha\beta}\,{}^{k}\Gamma^{*\delta}_{o\gamma}+\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{k}\Gamma^{*\delta}_{o\alpha}-\dot{\partial}_{\delta}g_{\gamma\alpha}\,{}^{k}\Gamma^{*\delta}_{o\beta}) \tag{2.5}$$

$k = 1,2,3,4.$

From (2.3), (2.4) and (2.5) it follows that

(2.6)

$$\text{(a)}\quad {}^{1}\Gamma^{*}_{0\beta\gamma} = \gamma_{0\beta\gamma} - 2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{1}\Gamma^{*\delta}_{00}$$

$$\text{(b)}\quad {}^{1}\Gamma^{*}_{0\beta 0} = \gamma_{0\beta 0}$$

$$\text{(c)}\quad {}^{2}\Gamma^{*}_{0\beta\gamma} = -2^{-1}(\lambda_{\gamma}\ell_{\beta}+\lambda_{0}g_{\beta\gamma}-\lambda_{\beta}\ell_{\gamma}) - 2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{2}\Gamma^{*\delta}_{00}$$

$$\text{(d)}\quad {}^{2}\Gamma^{*}_{0\beta 0} = -2^{-1}(2\lambda_{0}\ell_{\beta}-\lambda_{\beta})$$

$$\text{(e)}\quad {}^{3}\Gamma^{*}_{0\beta\gamma} = -2^{-1}(\lambda_{\gamma}\theta_{0\beta}+\lambda_{0}\theta_{\beta\gamma}-\lambda_{\beta}\theta_{\gamma 0}) - 2^{-1}L\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{3}\Gamma^{*\delta}_{00}$$

$$\text{(f)}\quad {}^{3}\Gamma^{*}_{0\beta 0} = -2^{-1}(2\lambda_{0}\theta_{0\beta}-\lambda_{\beta}\theta_{00})$$

$$\text{(g)}\quad {}^{4}\Gamma^{*}_{0\beta\gamma} = -2^{-1}(\tilde{\Gamma}_{0\beta\gamma}-\tilde{\Gamma}_{\beta\gamma 0}+\tilde{\Gamma}_{\gamma 0\beta}) - 2^{-1}L\dot{\partial}_{\delta}g_{\alpha\beta}\,{}^{4}\Gamma^{*\delta}_{00}$$

$$\text{(h)}\quad {}^{4}\Gamma^{*}_{0\beta 0} = \tilde{\Gamma}_{00\beta}.$$

From (1.18) and (2.2b) we obtain

$$A_{\alpha\beta\gamma} = {}^{1}A_{\alpha\beta\gamma}+{}^{2}A_{\alpha\beta\gamma}+{}^{3}A_{\alpha\beta\gamma}+{}^{4}A_{\alpha\beta\gamma}, \tag{2.7}$$

where

(2.8)

$$\text{(a)}\quad {}^{1}A_{\alpha\beta\gamma} = 2^{-1}L(\dot{\partial}_{\delta}g_{\alpha\beta}\delta^{\delta}_{\gamma}+\dot{\partial}_{\delta}g_{\beta\gamma}\delta^{\delta}_{\alpha}-\dot{\partial}_{\delta}g_{\gamma\alpha}\delta^{\delta}_{\beta}) - 2^{-1}T_{\alpha\beta\gamma}({}^{1}A)$$

$$\text{(b)}\quad {}^{2}A_{\alpha\beta\gamma} = -2^{-1}(\mu_{\gamma}g_{\alpha\beta}+\mu_{\alpha}g_{\beta\gamma}-\mu_{\beta}g_{\alpha\gamma}) - 2^{-1}T_{\alpha\beta\gamma}({}^{2}A)$$

$$\text{(c)}\quad {}^{3}A_{\alpha\beta\gamma} = -2^{-1}[((\mu_{\gamma}+2\ell_{\gamma})\theta_{\alpha\beta}+(\mu_{\alpha}+2\ell_{\alpha})\theta_{\beta\gamma}-(\mu_{\beta}+2\ell_{\beta})\theta_{\alpha\gamma}] - 2^{-1}T_{\alpha\beta\gamma}({}^{3}A)$$

$$\text{(d)}\quad {}^{4}A_{\alpha\beta\gamma} = 2^{-1}(\tilde{A}_{\alpha\beta\gamma}-\tilde{A}_{\beta\gamma\alpha}+\tilde{A}_{\gamma\alpha\beta}) - 2^{-1}T_{\alpha\beta\gamma}({}^{4}A)$$

$$T_{\alpha\beta\gamma}({}^{k}A) = L(\dot{\partial}_{\delta}g_{\alpha\beta}\,{}^{k}A^{\delta}_{0\gamma}+\dot{\partial}_{\delta}g_{\beta\gamma}\,{}^{k}A^{\delta}_{0\alpha}-\dot{\partial}_{\delta}g_{\gamma\alpha}\,{}^{k}A^{\delta}_{0\beta}) \tag{2.9}$$

$k = 1,2,3,4.$

From (2.7), (2.8) and (2.9) it follows that

(2.10)

(a) ${}^{1}A_{\sigma\beta\gamma} = 0 \Rightarrow$

(b) $T_{\alpha\beta\gamma}({}^{1}A) = 0$

(c) ${}^{2}A_{\sigma\beta\gamma} = -2^{-1}(\mu_\gamma \ell_\beta + \mu_o g_{\beta\gamma} - \mu_\beta \ell_\gamma) - 2^{-1} L\dot{\partial}_\delta g_{\beta\gamma}\, {}^{2}A^{\delta}_{oo} \Rightarrow$

(d) ${}^{2}A_{\sigma\beta o} = -2^{-1}(2\mu_o \ell_\beta - \mu_\beta)$

(e) ${}^{3}A_{o}\beta\gamma = -2^{-1}[(\mu_\gamma + 2\ell_\gamma)\theta_{\sigma\beta} + (\mu_o + 2)\theta_{\alpha\beta} - (\mu_\beta + 2\ell_\beta)\theta_{o\gamma}] -$

$2^{-1} L\dot{\partial}_\delta g_{\beta\gamma}\, {}^{3}A^{\delta}_{oo} \Rightarrow$

(f) ${}^{3}A_{\sigma\beta o} = -2^{-1}[2(\mu_o + 2)\theta_{o\beta} - (\mu_\beta + 2\ell_\beta)\theta_{oo}]$,

(g) ${}^{4}A_{\sigma\beta\gamma} = 2^{-1}(\tilde{A}_{\sigma\beta\gamma} - \tilde{A}_{\beta\gamma o} + \tilde{A}_{\gamma o\beta}) - 2^{-1} L\dot{\partial}_\delta g_{\beta\gamma}\, {}^{4}A^{\delta}_{oo} \Rightarrow$

(h) ${}^{4}A_{\sigma\beta o} = \tilde{A}_{oo\beta}$.

With respect to the coordinate transformation (1.5) and (1.6) ${}^{k}\Gamma^{*}_{\alpha\beta\gamma}$, $Q_{\alpha\beta\gamma}({}^{k}\Gamma)$, ${}^{k}\Gamma^{*}_{o\beta\gamma}$, ${}^{k}\Gamma^{*}_{o\beta o}$ for k = 2,3,4, $A_{\alpha\beta\gamma}$, ${}^{k}A_{\alpha\beta\gamma}$, $T_{\alpha\beta\gamma}({}^{k}A)$, ${}^{k}A_{o\beta\gamma}$, ${}^{k}A_{o\beta o}$ for k = 1,2,3,4 determined by (2.4b)-(2.5), (2.6c)-(2.6h) and (2.7)-(2.10) are transformed as tensors (because they are sums of tensors) and $\Gamma^{*\beta}_{\alpha\gamma}$, ${}^{1}\Gamma^{*\beta}_{\alpha\gamma}$, determined by (2.3), (2.4a) are transformed as connection coefficients (see 1.24) and (1.25)).

<u>Definition 2.1.</u> Recurrent Finsler space $F_N(D)$ supplied with the connection coefficients Γ^* and A which satisfy relations (2.3)-(2.10), will be denoted by $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_o,A_o)$.

<u>Lemma 2.2.</u> There are 2^5 essentially different types of connection coefficients in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_o,A_o)$.

<u>Proof.</u> If in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_o,A_o)$ some of vector fields λ, μ, or

some of tensor fields , $\tilde{A}$ or $\tilde{\Gamma}$ are zero, then in its place in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$ we shall put zero. The 2^5 different types of connection coefficients in $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$ are:

$$
\begin{array}{llll}
1. & F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A}) & 17. & F_N(D,0,0,0,\tilde{\Gamma},\tilde{A}) \\
2. & F_N(D,0,\mu,\theta,\tilde{\Gamma},\tilde{A}) & 18. & F_N(D,0,0,\theta,0,\tilde{A}) \\
3. & F_N(D,\lambda,0,\theta,\tilde{\Gamma},\tilde{A}) & 19. & F_N(D,0,\mu,0,0,\tilde{A}) \\
4. & F_N(D,\lambda,\mu,0,\tilde{\Gamma},\tilde{A}) & 20. & F_N(D,\lambda,0,0,0,\tilde{A}) \\
5. & F_N(D,\lambda,\mu,\theta,0,\tilde{A}) & 21. & F_N(D,0,0,\theta,\tilde{\Gamma},0) \\
6. & F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},0) & 22. & F_N(D,0,\mu,0,\tilde{\Gamma},0) \\
7. & F_N(D,0,0,\theta,\tilde{\Gamma},\tilde{A}) & 23. & F_N(D,\lambda,0,0,\tilde{\Gamma},0) \\
8. & F_N(D,0,\mu,0,\tilde{\Gamma},\tilde{A}) & 24. & F_N(D,0,\mu,\theta,0,0) \\
9. & F_N(D,0,\mu,\theta,0,\tilde{A}) & 25. & F_N(D,\lambda,0,\theta,0,0) \\
10. & F_N(D,0,\mu,\theta,\tilde{\Gamma},0) & 26. & F_N(D,\lambda,\mu,0,0,0) \\
11. & F_N(D,\lambda,0,0,\tilde{\Gamma},\tilde{A}) & 27. & F_N(D,\lambda,0,0,0,0) \\
12. & F_N(D,\lambda,0,\theta,0,\tilde{A}) & 28. & F_N(D,0,\mu,0,0,0) \\
13. & F_N(D,\lambda,0,\theta,\tilde{\Gamma},0) & 29. & F_N(D,0,0,\theta,0,0) \\
14. & F_N(D,\lambda,\mu,0,0,\tilde{A}) & 30. & F_N(D,0,0,0,\tilde{\Gamma},0) \\
15. & F_N(D,\lambda,\mu,0,\tilde{\Gamma},0) & 31. & F_N(D,0,0,0,0,\tilde{A}) \\
16. & F_N(D,\lambda,\mu,\theta,0,0) & 32. & F_N(D,0,0,0,0,0)
\end{array}
\tag{2.11}
$$

We shall use the notations:

$$ {}^k\Gamma = [({}^k\Gamma^*_{0\beta 0}=0)\wedge({}^k\Gamma^*_{0\beta\gamma}=0)\wedge({}^k\Gamma^*_{\alpha\beta\gamma}=0)] \quad k=2,3,4 \tag{2.12} $$

$$ {}^kA = [({}^kA_{0\beta 0}=0)\wedge({}^kA_{0\beta\gamma}=0)\wedge({}^kA_{\alpha\beta\gamma}=0)] \quad k=1,2,3,4 \tag{2.13} $$

$$
\begin{aligned}
{}^3\bar{A} = [& ({}^3A_{0\beta 0}=-(2\theta_{0\beta}-\ell_\beta\theta_{00})\wedge \\
& ({}^3A_{0\beta\gamma}=-(\ell_\gamma\theta_{0\beta}+\theta_{\gamma\beta}-\ell_\beta\theta_{0\gamma})-2^{-1}\partial_\delta g_{\beta\gamma}\,{}^3A^\delta_{00})\wedge \\
& ({}^3A_{\alpha\beta\gamma}=-(\ell_\gamma\theta_{\alpha\beta}+\ell_\alpha\theta_{\beta\gamma}-\ell_\beta\theta_{\alpha\gamma})-2^{-1}T_{\alpha\beta\gamma}({}^3A))]
\end{aligned}
\tag{2.14}
$$

Lemma 2.3. In the special cases of $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_\sigma,A_\sigma)$ given by (2.11) the connection coefficients which are different from (2,3)-(2.10) are denoted in the following list (where the notations from (2.12)-(2.14) are used):

1.
2. ${}^2\Gamma, {}^3\Gamma$
3. ${}^2A, {}^3\bar{A}$
4. ${}^3\Gamma, {}^3A$
5. ${}^4\Gamma$
6. ${}^4A$
7. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3\bar{A}$
8. ${}^2\Gamma, {}^3\Gamma, {}^3A$
9. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma$
10. ${}^2\Gamma, {}^3\Gamma, {}^4A$
11. ${}^3\Gamma, {}^2A, {}^3A$
12. ${}^4\Gamma, {}^2A, {}^3\bar{A}$
13. ${}^2A, {}^3\bar{A}, {}^4A$
14. ${}^3\Gamma, {}^4\Gamma, {}^3A$
15. ${}^3\Gamma, {}^3A, {}^4A$
16. ${}^4\Gamma, {}^4A$
17. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3A$
18. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3\bar{A}$
19. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^3A$
20. ${}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A$
21. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3\bar{A}, {}^4A$
22. ${}^2\Gamma, {}^3\Gamma, {}^3A, {}^4A$
23. ${}^3\Gamma, {}^2A, {}^3A, {}^4A$
24. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^4A$
25. ${}^4\Gamma, {}^2A, {}^3\bar{A}, {}^4A$
26. ${}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A, {}^4A$
27. ${}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A, {}^4A$
28. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^3A, {}^4A$
29. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3\bar{A}, {}^4A$
30. ${}^2\Gamma, {}^3\Gamma, {}^2A, {}^3A, {}^4A$
31. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A$
32. ${}^2\Gamma, {}^3\Gamma, {}^4\Gamma, {}^2A, {}^3A, {}^4A$.

Theorem 2.1. In the Finsler space there are 2^7 essentially different types of connection coefficients which by parallel displacement preserve the angle between two vectors.

Proof. As we proved in Theorem 1.2. in the space $F_N(D)$ the angle between two vectors by parallel displacement does not change. The connection coefficients in $F_N(D)$ which satisfy (1.2)-(1.4), (1.10)-(1.15), (1.20) depend on arbitrarily chosen seven parameters: λ_γ, $\mu_\gamma, \theta_{\alpha\beta}, \tilde{\Gamma}^\beta_{\alpha\gamma}, \tilde{A}^\beta_{\alpha\gamma}, N^\alpha_\beta, M^\alpha_\beta$. Instead of the arbitrary pair $(N^\alpha_\beta, M^\alpha_\beta)$, we can choose

$(L\Gamma^{*\alpha}_{\sigma\beta}, A^\alpha_{\sigma\beta}), (L\Gamma^{*\beta}_{\sigma\gamma}, 0)$, $(N^\alpha_\beta, 0)$, so we have

a. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(N,M)$

b. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$

c. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(N,0)$

d. $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})(\Gamma_0,0)$.

Since $F_N(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$ may take 2^5 different forms given in (2.11), so we have $4\cdot 2^5=2^7$ essentially different kinds of connection coefficients which depend on the choice of (N,M), $(\Gamma_0,A_0)(N,0)$ or $(\Gamma_0,0)$.

It is clear from the above that all the mentioned connection coefficients are special cases of a.1. The first introduced and most examined case is d.32. which is $F_N(D,0,0,0,0,0)(\Gamma_0,0)$. It is the Finsler space supplied with the well known Cartan connection coefficients. We obtain them from (1.17) and (1.18), if we put them in

$$\lambda_\gamma=0,\mu_\gamma=0,\theta_{\alpha\beta}=0,\tilde{\Gamma}_{\alpha\beta\gamma}=0,\tilde{A}_{\alpha\beta\gamma}=0,N^\alpha_\beta=L\Gamma^\alpha_{0\beta},M^\alpha_\beta=0.$$

These connection coefficients appear in the papers of E.Cartan, O.Varga, in the book by H.Rund [9] and many other authors.

Parameters λ_γ and μ_γ were first introduced by A.Moór in |6|. He examined special cases of b.4. i.e. such a space

$F_N(D,\lambda,\mu,0,\tilde{\Gamma},\tilde{A})(\Gamma_0,A_0)$ in which $\Gamma^*_{\alpha\beta\gamma}=\Gamma^*_{\beta\alpha\gamma}$ and $A_{\alpha\beta\gamma}=A_{\beta\alpha\gamma}$.

The nonlinear connection N^α_β has recently been introduced and appears in a book by M.Matsumoto [4]. He studied in |5| the space c.17., the space $F_N(D,0,0,0,\Gamma,A)(N,0)$ in which $\lambda_\gamma=0,\mu_\gamma=0,\theta_{\alpha\beta}=0$ and $M^\alpha_\beta=0$.

Parameter θ was first introduced by the present author and some special cases of b. were studied in [1], [2].

In this paper the tensor M^α_β is introduced for the first time, and the most general till now known connection coefficients are obtained by (1.17) and (1.18).

§ 3. SOME FUNDAMENTAL RELATIONS IN F_N $(D,\lambda,\mu,\theta,\tilde{\Gamma},\tilde{A})$ (N,M)

From

$$g_{\alpha\beta}g^{\beta\delta}=\delta^{\delta}_{\alpha},$$

we have

(3.1) $$g_{\alpha\beta|\gamma}g^{\beta\delta}+g^{\beta\delta}{}_{|\gamma}g_{\alpha\beta}=0$$

(3.2) $$g_{\alpha\beta}|_{\gamma}\,g^{\beta\delta}+g^{\beta\delta}|_{\gamma}g_{\alpha\beta}=0.$$

From (3.1) and (1.22) we get

(3.3) $$g^{\beta\delta}{}_{|\gamma}=-\lambda_{\gamma}(g^{\beta\delta}+\theta^{\beta\delta})\quad -2\theta^{\beta\delta}(\Gamma^{*o}_{o\gamma}-L^{-1}N^{o}_{\gamma}).$$

From (3.2) and (1.23) we have

(3.4) $$g^{\beta\delta}|_{\gamma}=-\mu_{\gamma}g^{\beta\delta}-(\mu_{\gamma}+2\ell_{\gamma})\theta^{\beta\delta}\quad -2\theta^{\beta\delta}(A^{o}_{o\gamma}-M^{o}_{\gamma}).$$

In (3.3) and in (3.4)

$$\theta^{\beta\delta}=g^{\beta\kappa}g^{\delta\rho}\theta_{\kappa\rho}.$$

From (3.3), (3.4) and (1.26) we have

(3.5) $$Dg^{\beta\delta}=g^{\beta\delta}{}_{|\gamma}dx^{\gamma}+g^{\beta\delta}|_{\gamma}\bar{D}\ell^{\gamma}=-(\lambda_{\gamma}dx^{\gamma}+\mu_{\gamma}\bar{D}\ell^{\gamma})g^{\beta\delta}.$$

By direct calculation,we obtain

(3.6) $$\dot{x}^{\delta}{}_{|\gamma}=\partial_{\gamma}\dot{x}^{\delta}-(\dot{\partial}_{\kappa}\dot{x}^{\delta})N^{\kappa}_{\gamma}+\Gamma^{*}{}_{\kappa}{}^{\delta}{}_{\gamma}\dot{x}^{\kappa}=\Gamma^{*\delta}_{\gamma}-N^{\delta}_{\gamma},\quad (\Gamma^{*\delta}_{\gamma}=\Gamma^{*\delta}_{\beta\gamma}\dot{x}^{\beta})$$

Introducing the notation $I^{\kappa}_{\gamma}=\delta^{\kappa}_{\gamma}-M^{\kappa}_{\gamma}$, we have

(3.7) $$\dot{x}^{\delta}|_{\gamma}=L(\dot{\partial}_{\kappa}\dot{x}^{\delta})I^{\kappa}_{\gamma}+A_{\kappa}{}^{\delta}{}_{\gamma}\dot{x}^{\kappa}=LI^{\delta}_{\gamma}+A_{\kappa}{}^{\delta}{}_{\gamma}\dot{x}^{\kappa}.$$

If we put

(3.8) $$\ell_{\delta}|_{\gamma}=L(\dot{\partial}_{\iota}\ell_{\delta})I^{\iota}_{\gamma}-A_{\delta}{}^{\kappa}{}_{\gamma}\ell_{\kappa},$$

then from

$$L\dot{\partial}_\iota \ell_\delta = L\dot{\partial}_\iota \dot{\partial}_\delta L = 2^{-1}\dot{\partial}_\iota \dot{\partial}_\delta L^2 - \dot{\partial} L \dot{\partial}_\delta L$$

and (3.8) we obtain

$$\ell_\delta|_\gamma = (g_{\iota\delta} - \ell_\iota \ell_\delta) I^\iota_\gamma - A_\delta{}^\kappa{}_\gamma \ell_\kappa . \tag{3.9}$$

For $\ell^\delta|_\gamma$ we have

$$\ell^\delta|_\gamma = (g^{\delta\kappa}\ell_\kappa)|_\gamma = g^{\delta\kappa}|_\gamma \ell_\kappa + g^{\delta\kappa}\ell_\kappa|_\gamma .$$

Substituting (3.4) and (3.9) into the above equation, we obtain

$$\ell^\delta|\gamma = -\mu_\gamma \ell^\delta - (\mu_\gamma + 2\ell_\gamma)\theta^{o\delta} + (\delta^\delta_\iota - \ell_\iota \ell^\delta) I^\iota_\gamma - g^{\delta\kappa} A_\kappa{}^o{}_\gamma - 2(A^o_{o\gamma} - M^o_\gamma)\theta^{o\delta}. \tag{3.10}$$

For any scalar function $G(x,\dot{x})$, we shall introduce the notations

$$G_{|\gamma} = \partial_\gamma G - (\dot{\partial}_\iota G) N^\iota_\gamma \tag{3.11}$$

$$G|_\gamma = L(\dot{\partial}_\kappa G) I^\kappa_\gamma . \tag{3.12}$$

We have

$$DG = dG = \partial_\gamma G dx^\gamma + \dot{\partial}_\gamma G d\dot{x}^\gamma.$$

Substituting $d\dot{x}^\iota$ from (1.4) we get

$$dG = \partial_\gamma G dx^\gamma + L(\dot{\partial}_\iota G) I^\iota_\gamma \bar{D}\ell^\gamma - L(\dot{\partial}_\iota G)\dot{x}^\iota \, dL^{-1} - (\dot{\partial}_\iota G) N^\iota_\gamma dx^\gamma , \tag{3.13}$$

from which it follows that

$$DG = dG = G_{|\gamma} dx^\gamma + G|_\gamma \bar{D}\ell^\gamma$$

is true only when $G(x,\dot{x})$ is homogeneous to degree zero in $\dot{x}$.
For $G(x,\dot{x}) = L(x,\dot{x})$ we have

$$L_{|\gamma} = \partial_\gamma L - (\dot{\partial}_\iota L) N^\iota_\gamma = \partial_\gamma L - N^o_\gamma \tag{3.14}$$

$$L|_\gamma = L(\dot{\partial}_\kappa L) I^\kappa_\gamma = L(\ell_\gamma - M^o_\gamma). \tag{3.15}$$

Since $L(x,\dot{x})$ is homogeneous to degree one in $\dot{x}$, then from (3.13) follows

$$dL = L_{|\gamma}dx^{\gamma} + L|_{\gamma}\bar{D}\ell^{\gamma} + dL.$$

From the above equation, we obtain

(3.16) $$L_{|\gamma}dx^{\gamma} + L|_{\gamma}\bar{D}\ell^{\gamma} = 0.$$

From

(3.17) $$L^{2} = g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta},$$

we obtain

$$2LL_{|\gamma} = g_{\alpha\beta|\gamma}\dot{x}^{\alpha}\dot{x}^{\beta} + 2g_{\alpha\beta}\dot{x}^{\alpha}{}_{|\gamma}\dot{x}^{\beta}.$$

Substituting (1.22) and (3.6) into above equation, we obtain

(3.18) $$L_{|\gamma} = 2^{-1}L\lambda_{\gamma}(1+\theta_{oo}) + \ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})(1+\theta_{oo}).$$

In a similar way from (3.17) using (1.23) and (3.7), we get

(3.19) $$L|_{\gamma} = 2^{-1}L\mu_{\gamma} + 2^{-1}L(\mu_{\gamma}+2\ell_{\gamma})\theta_{oo} + L\ell_{\gamma} + L(A_{o}{}^{o}{}_{\gamma} - M^{o}_{\gamma})(1+\theta_{oo}).$$

From

$$\dot{x}^{\delta}{}_{|\gamma} = L_{|\gamma}\ell^{\delta} + L\ell^{\delta}{}_{|\gamma}$$

(3.6) and (3.18), we obtain

$$\ell^{\delta}{}_{|\gamma} = L^{-1}(\Gamma^{*\delta}_{\gamma} - N^{\delta}_{\gamma}) - \ell^{\delta}2^{-1}\lambda_{\gamma}(1+\theta_{oo}) - L^{-1}\ell^{\delta}\ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})(1+\theta_{oo}),$$

i.e.

(3.20) $$\ell^{\delta}{}_{|\gamma} = L^{-1}(\delta^{\delta}_{\alpha} - \ell^{\delta}\ell_{\alpha})(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma}) - 2^{-1}\ell^{\delta}\lambda_{\gamma}(1+\theta_{oo}) - L^{-1}\ell^{\delta}\ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})\theta_{oo}.$$

Using (1.22) and (3.20) we have

$$\ell_{\delta|\gamma} = (g_{\delta\kappa}\ell^{\kappa})_{|\gamma} = g_{\delta\kappa|\gamma}\ell^{\kappa} + g_{\delta\kappa}\ell^{\kappa}{}_{|\gamma}$$

(3.21) $$\ell_{\delta|\gamma} = \lambda_{\gamma}\ell_{\delta} + \lambda_{\gamma}\theta_{o\delta} - 2^{-1}\ell_{\delta}\lambda_{\gamma}(1+\theta_{oo}) + L^{-1}(g_{\delta\alpha} - \ell_{\delta}\ell_{\alpha})(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})$$

$$+L^{-1}(-\theta_{oo}\ell_{\delta}\ell_{\alpha} + 2\theta_{o\delta}\ell_{\alpha})(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})$$

Relations (3.14) and (3.18), further more (3.15) and (3.19) are consistent, which can be proved in the following way: From (1.17) we obtain

(3.22) $$\Gamma^{*}_{oo\gamma} = \Gamma^{*}_{\alpha\beta\gamma}\ell^{\alpha}\ell^{\beta} = \gamma_{oo\gamma} - 2^{-1}\lambda_{\gamma}(1+\theta_{oo}) - \theta_{oo}(\Gamma^{*o}_{o\gamma} - L^{-1}N^{o}_{\gamma}),$$

because

$$\tilde{\Gamma}_{\gamma oo} - \tilde{\Gamma}_{o\gamma o} + \tilde{\Gamma}_{\gamma oo} = 0.$$

From (1.1) we have

$$\gamma_{oo\gamma} = \gamma_{\alpha\beta\gamma}\ell^{\alpha}\ell^{\beta} =$$

$$2^{-1}L^{-2}[\partial_{\gamma}\dot{\partial}_{\alpha}\dot{\partial}_{\beta}(2^{-1}L^{2}) + \partial_{\alpha}\dot{\partial}_{\beta}\dot{\partial}_{\gamma}(2^{-1}L^{2}) - \partial_{\beta}\dot{\partial}_{\alpha}\dot{\partial}_{\gamma}(2^{-1}L^{2})]\dot{x}^{\alpha}\dot{x}^{\beta}.$$

Using the homogeneity condition, we obtain

(3.23) $$\gamma_{oo\gamma} = L^{-1}\partial_{\gamma}L.$$

Using the relation

$$\ell_{\alpha}\Gamma^{*\alpha}_{\gamma} = L\Gamma^{*}_{oo\gamma},$$

further substituting (3.22) and (3.23) into (3.18), we obtain (3.14).

On the other side from (1.18), we have

(3.24) $$(1+\theta_{oo})(A^{o}_{o\gamma} - M^{o}_{\gamma}) = -2^{-1}\mu_{\gamma} - 2^{-1}(\mu_{\gamma} + 2\ell_{\gamma})\theta_{oo} - M^{o}_{\gamma}.$$

Substituting (3.24) into (3.19), we obtain (3.15).

From (3.18) and (3.19) we get

(3.25) $$L_{|\gamma}dx^{\gamma} + L|_{\gamma}\bar{D}\ell^{\gamma} = 2^{-1}L(1+\theta_{oo})[\lambda_{\gamma} + L^{-1}2\ell_{\alpha}(\Gamma^{*\alpha}_{\gamma} - N^{\alpha}_{\gamma})]dx^{\gamma} +$$

$$2^{-1}L(1+\theta_{oo})[\mu_{\gamma} + 2\ell_{\gamma} + 2(A^{o}_{o\gamma} - M^{o}_{\gamma})]\bar{D}\ell^{\gamma},$$

Using (1.20) we obtain

(3.26) $L_{|\gamma} dx^{\gamma} + L_{|\gamma} \bar{D}\ell^{\gamma} = 0.$

In the space d. 32. i.e. in $F_N(D,0,0,0,0,0)(\Gamma_o,0)$ where $\lambda_{\gamma} = 0$, $\mu_{\gamma} = 0$, $\theta_{\alpha\beta} = 0$, $\tilde{\Gamma}^{*}_{\alpha\beta\gamma} = 0$, $\tilde{A}_{\alpha\beta\gamma} = 0$, $N^{\alpha}_{\beta} = \Gamma^{*\alpha}_{\beta} = L\Gamma^{*\alpha}_{o\beta}$, $M^{\alpha}_{\beta} = 0$ $I^{\kappa}_{\gamma} = \delta^{\kappa}_{\gamma}$, $A_o{}^{\kappa}{}_{\beta} = 0$, $A_{\alpha}{}^{o}{}_{\beta} = 0$, $A_o{}^{\alpha}{}_o = 0$, we have from (3.6), (3.7), (3.9), (3.10), (3.18), (3.19), (3.20), (3.21) respectively

$$\dot{x}^{\delta}{}_{|\gamma} = 0, \qquad \dot{x}^{\delta}{}_{|\gamma} = L\delta^{\delta}_{\gamma}$$

$$\ell_{\delta|\gamma} = g_{\delta\gamma} - \ell_{\delta}\ell_{\gamma}$$

$$\ell^{\delta}{}_{|\gamma} = \delta^{\delta}_{\gamma} - \ell^{\delta}\ell_{\gamma}$$

$$L_{|\gamma} = 0, \quad L_{|\gamma} = L\ell_{\gamma}, \qquad \ell^{\delta}{}_{|\gamma} = 0, \quad \ell_{\delta|\gamma} = 0,$$

which are the well known relations in the Finsler space supplied by the Cartan connection coefficients.

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REZIME

AFINE KONEKCIJE U FINSLEROVOM PROSTORU

U Finslerovom prostoru su uvedeni generalisani koeficijenti koneksije koji zavise od sedam proizvoljno izabranih parametara. Specijalni slučajevi ovih su Cartanovi koeficijenti koneksije, kao i oni iz rekurentnih Finslerovih prostora. Svi imaju osobinu da se pri paralelnom pomeranju ugao izmedju vektora ne menja. Pokazano je da postoji 2^7 bitno različitih tipova koneksija ove vrste. Za generalni slučaj su nadjene neke fundamentalne relacije.

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