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A NUMERICAL SOLUTION OF THE SINGULAR PERTURBATION PROBLEM ARISING FROM A WEAKLY COUPLED SYSTEM

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Abstract

A first order differential equation with a small perturbation parameter is derived from a weakly coupled first order system. It is solved numerically by the finite-difference method on a special discretization mesh.

1. Introduction

Let us consider the following system of first order differential equations:

(1.a) $$y' = f(x) \;,$$

(1.b) $$\varepsilon u' + a(x,u) = 0, \quad x \in I = [0,1] \;,$$

(1.c) $$y(0) + b_0 u(0) = c_0 \;,$$

(1.d) $$y(1) + b_1 u(1) = c_1 \;,$$

where ε is a small perturbation parameter, $0 < \varepsilon << 1$; $b_i, c_i \in R$, $i = 0,1$, $b_0 b_1 < 0$; f and a are sufficiently smooth functions in I and $I \times R$, respectively, and $a_u(x,u) > a_* > 0$, $x \in I$, $u \in R$.

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The physical meaning of problem (1) can be found in [7], where the linear case was considered. An exponentially fitted higher order scheme was introduced there and the convergence uniform in ε was proved. Here we propose a simpler approach which decouples y and u from (1) in the following way.

Let $g(x)$ be the solution to (1.a) satisfying the condition $g(0)=0$. Then we have

$$y(x) = g(x) + d \,, \tag{2}$$

where $d = y(0)$. Then the boundary conditions (1.c,d) reduce to

$$\begin{aligned} b_0 u(0) &= c_0 - d \,, \\ b_1 u(1) &= c_1 - g(1) - d \,. \end{aligned} \tag{3}$$

By substracting these equations we eliminate d and get:

$$u(0) + bu(1) = c \,, \tag{4}$$

with the appropriate $b > 0$ and c. It is easy to evaluate $g(1)$ and thus to get c. Then problem (1.b), (4) can be solved numerically. By using the numerical approximation of $u(0)$ we can get d from (3) and find $y(x)$ from (2).

Thus we can consider problems of the type (1.b), (4) only. We shall solve them numerically using the classical finite-difference schemes on special discretization meshes. By the standard technique we can prove the first order convergence uniform in ε (see [1,2,8,9,10] for instance).

The solution to problem (1.b), (4) has a layer of width $O(\varepsilon)$ at the origin. Our discretization mesh is generated by a suitable function which maps an equidistant mesh to the mesh which is dense in the layer. The density changes automatically when ε does. Because of that we obtain uniform numerical results within the region where the continuous solution changes abruptly.

We note that the numerical methods for more general systems have been considered in a number of papers, let us mention [6] and [11] only. However, because of the special structure of system (1) we can construct here a very simple and efficient method.

Throughout the paper we denote by M any positive constant which is independent of ε and of the discretization mesh.

2. The Continuous Problem

Thus, we consider the problem:

$$(5)\qquad \begin{aligned} Tu &:= \varepsilon u' + a(x,u) = 0\,, \quad x \in I\,,\\ B_b u &:= u(0) + bu(1) = c\,, \end{aligned}$$

$$0 < \varepsilon \le \varepsilon_0,\quad b \ge 0,\quad a \in C^1(I \times R),\quad a_u(x,u) > a_*>0,\quad (x,u) \in I \times R.$$

Let us denote by $u_b \in C^2(I)$ the solution to problem (5).

The case $b=0$ is well known. Operator (T,B_0) is inverse monotone and the solution u_0 exists uniquely. Using the technique from [1,8] (cf. [2] as well) we can prove (the details will be omitted):

Theorem 1. *For the solution u_0 to problem* (5) *with $b=0$, we have*

$$|u_0'(x)| \le M(1 + \varepsilon^{-1}v(x))\,,$$

$$|u_0''(x)| \le M(\varepsilon^{-1}+ \varepsilon^{-2}v(x)),$$

where $x \in I$ *and* $v(x) = \exp(-a_* x/\varepsilon)$.

Now we can prove

Theorem 2. *The solution u_b to problem* (5) *with $b > 0$ uniquely exists and its derivatives satisfy the same estimates as the derivatives of u_0 in Theorem* 1.

Proof. Let $z \in C^1(I)$ denote the unique solution to the reduced problem

$$a(x,z) = 0$$

and let

$$|z(x)| \le Z\,,\quad x \in I\,.$$

Then, by the inverse monotonicity of (T,B_0) we can get that the solution u_0 to problem

$$Tu = 0\,,\quad B_0 u = s,$$

satisfies

$$P:=\min(s,-Z) \le u_0 \le \max(s,Z) =:Q\,.$$

Let us consider two values of s :

$$s_1 < \min(0,c-bZ)\,,$$

$$s_2 > \max(0,c+bZ)\,.$$

We have the corresponding solutions $u_{0,i}$ and constants P_i, Q_i, $i=1,2$. Thus we get

$$Tu_{0,i} = 0, \quad i=1,2\ ,$$

$$B_b u_{0,1} \le s_1 + bQ_1 = s_1 + bZ < c$$

and similarly

$$B_b u_{0,2} > c\ .$$

Because of the fact that u_0 depends continuously on s, we can conclude that there exists a unique value s_*, such that the solution $u_{0,*}$ to problem

$$Tu = 0\ , \quad B_0 u = s_*\ ,$$

satisfies

$$B_b u_{0,*} = c\ .$$

Hence $u_{0,*} = u_b$ and we have the same estimates as in Theorem 1. The Theorem is proved.

We can see that u_b has a layer of width $O(\varepsilon)$ at $x=0$. The estimates of the derivatives of u_b are important for the proof of the convergence uniform in ε.

3. The Discretization

Let us use mesh I_h from [9], cf. [1,2,8,10]:

$$x_i = m(ih)\ , \quad i=0,1,\dots,n\ , \quad h=1/n\ , \quad n \in N\ ,$$

$$m(t) = \begin{cases} F(t) := A\varepsilon t/(q-t)\ , & t \in [0,p] \\ F(p) + F'(p)(t-p)\ , & t \in [p,1] \end{cases}$$

Here $q \in (0,1)$ and $A \in (0,q/\varepsilon_0)$ are fixed numbers and $p \in (0,q)$ is the abscissa of the contact point of tangent line from $(1,1)$ to the curve $F(t)$. The point p can be found explicitly.

We form the discretization of problem (5) using finite difference scheme on mesh I_h:

(6a) $$w_0 + bw_n = c\ ,$$

(6b) $$T_h w_i := \varepsilon D\_w_i + a(x_i, w_i) = 0\ , \quad i=1,2,\dots,n\ ,$$

where $\{w_i\}$ denotes a mesh function on I_h and

$$D\_w_i = (w_i - w_{i-1})/h_i\ ,$$

$$h_i = x_i - x_{i-1} , \quad i=1,2,\dots,n .$$

The existence and uniqueness of the solution to the discrete problem (3) can be established in the same way as in the continuous case.

Theorem 3. *Let* $\{w_i\}$ *be the solution to the discrete problem* (6) *on the mesh* I_h *with* $n > 2/q$ *and let* u_b *be the solution to the continuous problem* (5). *Then we have*

$$|w_i - u_b(x_i)| \le M\, h .$$

Proof. By using the technique from [8,1] we can easily prove the consistency uniform in ε, i.e.:

$$|T_h u_b(x_i) - T_h w_i| =$$
$$|T_h u_b(x_i) - (Tu_b)(x_i)| =$$
$$\varepsilon|D_- u_b(x_i) - u'(x_i)| \le M\, h , \quad i=1,2,\dots,n .$$

There remains to prove the stability uniform in ε. Let us first consider the linear case, i.e. problem (5) when $a(x,u) = a(x)u - r(x)$, $a(x) > a_* > 0$, $x \in I$. Its discretization on mesh I_h reads:

$$w_0 + bw_n = c ,$$
$$w_i = A_i w_{i-1} + B_i , \quad i=1,2,\dots,n ,$$

where

$$A_i = \varepsilon/(\varepsilon + a(x_i)h_i) , \quad B_i = A_i r(x_i)h_i/\varepsilon .$$

We shall prove the stability inequality:

(7) $$|w_i| \le M(|c| + R) ,$$

where $|r(x_i)| \le R$, $i=1,2,\dots n$. For the technique cf. [4]. We have

$$w_i = K_i w_0 + L_i , \quad i=1,2,\dots,n ,$$
$$K_i = A_i K_{i-1} , \quad i=1,2,\dots,n , \ K_0 = 1 ,$$
$$L_i = A_i L_{i-1} + B_i , \quad i=1,2,\dots,n , \ L_0 = 0 ,$$
$$w_0 = (c - bL_n)/(1 - bK_n) .$$

Since $0 < K_i < 1$, $i=1,2,\dots,n$, inequality (7) follows if we show

(8) $$|L_i| \le M\, R , \quad i=1,2,\dots,n .$$

It holds that

(9) $$|L_i| \le L'_i\ ,\quad i=1,2,\dots,n\ ,$$

where

$$L'_i = A'_i L'_{i-1} + B'_i\ ,\quad i=1,2,\dots,n\ ,\quad L'_0 = 0\ ,$$
$$A'_i = \varepsilon/(\varepsilon + a_* h_i) \ge A_i\ ,$$
$$B'_i = A'_i R h_i/\varepsilon \ge |B_i|\ .$$

By induction we can prove :

(10) $$L'_i \le R/a_*\ ,\quad i=1,2,\dots,n\ ,$$

and because of (9) we get (8), and (7) as well. Indeed, it is easy to verify that (10) holds for $i=1$. Now suppose that (10) is valid for some i. We have

$$L'_{i+1} \le A'_i R/a_* + B'_i =$$
$$= A'_i R(1/a_* + h_i/\varepsilon) = R/a_*\ .$$

In the nonlinear case we have

$$T_h u_b(x_i) - T_h w_i =$$
$$\varepsilon D\_(u_b(x_i) - w_i) + Q_i(u_b(x_i) - w_i)\ ,$$

where

$$Q_i = \int_0^1 a_u(x_i, w_i + s(u_b(x_i) - w_i))ds > a_* > 0\ .$$

Hence, the linear convergence uniform in ε follows in the same way as in the linear case. The Theorem is proved.

4. Numerical Examples

We shall consider two problems from [3], modified to the form of our problem (5). The first problem is linear:

(11) $$\varepsilon u' + u = r(x) + \varepsilon r'(x)\ ,\quad u(0) + u(1) = c\ ,$$

where

$$r(x) = 10 - (10 + x)\exp(-x)\ ,$$
$$c = r(1) + 10(1 + \exp(-1/\varepsilon))\ ,$$

so that the solution reads $u = r(x) + 10\exp(-x/\varepsilon)$.

The second problem is

$$\varepsilon u' + u^2 = 0 ,$$

(12)

$$u(0) + u(1) = 10 + 1/(0.1 + 1/\varepsilon) .$$

Its solution reads $u = 1/(0.1 + x/\varepsilon)$.

In all the numerical experiments we use the mesh generating function λ with parameters A=1, q=0.9, thus obtaining about 45 % of the mesh points in the interval $[0,\varepsilon]$ which represents the layer.

Let us denote by E the maximum pointwise error. We have achieved the same values of E for all ε's which were considered: $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}$. In the case of problem (11), we have E=0.126 for n=50 and E=0.0642 for n=100. In the case of problem (12), we have E=0.199 for n=100 and E=0.102 for n=200.

The nonlinear system (6) was solved by the Newton-Kantorovich method (see [5] for instance). The initial approximation was a constant mesh-function $c/(1 + b)$. The iterations were carried on until the maximal pointwise difference between two succesive iterations became less than 0.01. Nine iterations were needed.

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Rezime

NUMERIČKO REŠAVANJE SINGULARNO PERTURBOVANOG PROBLEMA KOJI PROIZILAZI IZ JEDNOG SLABO POVEZANOG SISTEMA

Diferencijalna jednačina prvog reda sa malim parametrom je izvedena iz jednog slabo povezanog sistema prvog reda. Zatim je rešavana numerički pomoću metoda konačnih razlika na specijalnoj mreži diskretizacije.

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