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A NOTE ON SOME LATTICE CHARACTERIZATIONS OF HAMILTONIAN GROUPS

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ABSTRACT

It is shown that the lattice of all the congruences on all the subgroups (i.e. the lattice of all weak congruences) of a group is modular if and only if the group is Hamiltonian (this is the solution of a problem stated in [4]). It is also proved that a group is Hamiltonian if and only if its diagonal relation is an exceptional element in the above-mentioned lattice.

A *weak congruence relation* ([3]) on an algebra $\mathcal{A}=(A,F)$ is a symmetric, transitive and compatible relation ρ on A, satisfying a weak reflexivity: if c is a constant in $\mathcal{A}$, then

$c\rho c$.

For a given algebra $\mathcal{A}$, let $S(\mathcal{A})$, $C(\mathcal{A})$, and $C_W(\mathcal{A})$

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be the lattices of subalgebras, congruences and weak congruences on $\mathcal{A}$, respectively.

It is clear that $C_w(\mathcal{A})$ is an algebraic lattice under set inclusion and that it coincides with the lattice of all the congruences on all the subalgebras of $\mathcal{A}$.

$C(\mathcal{A})$ is a sublattice of $C_w(\mathcal{A})$, namely, it is a filter $[\Delta)$, where $\Delta=\{(x,x) \mid x\in A\}$ (diagonal relation on $\mathcal{A}$).

$S(\mathcal{A})$ is isomorphic with the ideal $(\Delta]$, under the mapping $B\to d_{B^2}$, where $d_{B^2}=\{(x,x) \mid x\in B\}$ (a diagonal relation on $B\in S(\mathcal{A})$).

Moreover, $S(\mathcal{A})$ is a homomorphic image of $C_w(\mathcal{A})$, under $\rho\to B$, where $d_\rho = d_{B^2}$, and $d\rho \overset{def}{=} \rho\wedge\Delta$ (a diagonal of ρ).

In the following, we shall identify the subalgebras of $\mathcal{A}$ with the corresponding diagonal relations in $C_w(\mathcal{A})$.

$\mathcal{A}$ is said to have the *congruence intersection property* (CIP) ([3]) if for all $\rho,\theta\in C_w(\mathcal{A})$

$$(\rho\wedge\theta)_A = \rho_A \wedge \theta_A,$$

where $\rho_A \overset{def}{=} \bigcap(\sigma\in C(\mathcal{A}) \mid \rho\subseteq\sigma)$.

Obviously, $\rho_A = \rho\vee\Delta$, and thus CIP expresses a distributivity property of Δ.

Recall that $\mathcal{A}$ satisfies the *congruence extension property* (CEP) if every congruence on an arbitrary subalgebra of $\mathcal{A}$ is a restriction of some congruence on $\mathcal{A}$.

It was proved in [3] that:

(I) An algebra $\mathcal{A}$ has a modular lattice of weak congruences if and only if $\mathcal{A}$ satisfies CEP and CIP, and both $S(\mathcal{A})$ and $C(\mathcal{A})$ are modular lattices.

It was proved in [5] that:

(II) $\mathcal{A}$ satisfies CEP and CIP if and only if the mapping $\rho\to(d\rho,\rho_A)$ is an embedding from $C_w(\mathcal{A})$ into $S(\mathcal{A})\times C(\mathcal{A})$.

If G is a group, then every $\rho \in C_w(G)$ uniquely determines a pair (H,K) of subgroups of G, where $K \triangleleft H$, $K=[e]_\rho$ (an equivalence class containing a neutral element e - constant in G), and H is represented by a diagonal relation d_ρ, as mentioned before. Conversely, every pair (H,K) of subgroups of G, such that K is normal in H, determines one weak congruence of G, namely the congruence on H which corresponds to $K \triangleleft H$. Thus, the lattice $C_w(G)$ is isomorphic with the lattice of all pairs (H,K), where H runs over all the subgroups of G, and K over the normal subgroups of H.

Now let $\bar{K}$ be a least normal subgroup of G containing a subgroup K i.e. let

$$\bar{K} \overset{def}{=} \cap (K_i \triangleleft G \mid K < K_i).$$

PROPOSTITION 1. *A group* G *satisfies* CEP *and* CIP *if and only if* $K \to \bar{K}$ *is an embedding from* S(G) *into* C(G) (the latter considered as a lattice of all the normal subgroups of G).

Proof.

By the above-mentioned isomorphism between $C_w(G)$ and the lattice of ordered pairs of subgroups, the embedding from (II) can be given by $(H,\bar{K}) \to (H,\bar{K})$, $K \triangleleft H < G$, and hence by $K \to \bar{K}$. □

COROLLARY 2. *A group* G *satisfies* CEP *and* CIP *if and only if it is Hamiltonian.*

Proof.

If G satisfies CEP and CIP, then the embedding $K \to \bar{K}$ from S(G) into C(G) maps the normal subgroup into itself. Thus, S(G)=C(G), and G is Hamiltonian. The converse follows from Proposition 1. □

Corollary 2., together with (I), yields the following.

THEOREM 3. *A group is Hamiltonian if and only if its lattice of weak congruences is modular.*□

PROBLEM . Characterize groups satisfying CIP, which in this case has the following form for any two subgroups H,K of G,

$$\overline{H\cap K} = \overline{H}\cap\overline{K} .$$

(If G is finite, then CIP is equivalent with being Hamiltonian, see [4]).

Some other characterization of Hamiltonian groups can be given by means of the neutral and exceptional elements of the lattice $C_w(G)$.

Recall that an element a of a bounded lattice L (with 0 and 1) is said to be *neutral* if it satisfies the identity

$$(a\wedge x)\vee(x\wedge y)\vee(y\wedge a)=(a\vee x)\wedge(x\vee y)\wedge(y\vee a),$$

for all $x,y\in L$.

One can prove that:

a) a is neutral in L if and only if the mappings

$$m_a: x\to x\wedge a \quad \text{and} \quad n_a: x\to x\vee a \quad (x\in L)$$

are homomorphisms and

$$f_a: x\to (x\wedge a, x\vee a)$$

is an embedding from L into (a]x[a).

An element a of a bounded lattice L is *exceptional* if it is neutral and the classes of the congruence induced by m_a have maximum elements which form a sublattice of L(see[2]).

Now, if we consider the bounded lattice $C_w(A)$ of an algebra A, and the diagonal relation $\Delta\in C_w(A)$, then we have the following statements, the proofs of which are straightforward:

a) $m_\Delta : \rho \to \rho \wedge \Delta$ ($\rho \in C_w(A)$) is a lattice homomorphism (a homomorphic image is $S(A)$, as already stated).

2) $n_\Delta : \rho \to \rho \vee \Delta$ ($\rho \in C_w(A)$) is a lattice homomorphism if and only if A satisfies CIP.

3) Δ is a neutral element of the lattice $C_w(A)$ if and only if A satisfies CEP and CIP (see[4]).

4) The classes induced by the congruence m_Δ in $C_w(A)$ have maximal elements; the set of such elements is

$M_\Delta = \{B^2 \mid B < A\}$.

(M_Δ is not necessarily a sublattice of $C_w(A)$).

In the case of groups, this yields the following:

LEMMA 4. *If G is a group, then* $M_\Delta = \{H^2 \mid H < G\}$ *is a sublattice of* $C_w(G)$.

<u>Proof</u>.

Since $H^2 \cap K^2 = (H \cap K)^2$, for the subgroups H,K of G, all we have to prove is that in the lattice $C_w(G)$

$(H \vee K)^2 \leq H^2 \vee K^2$

(since obviously $H^2 \vee K^2 < (H \vee K)^2$).

If e is a neutral element of G, then

$(H \vee K)^2 = ([e]_{(H \vee K)^2})^2 \leq ([e]_{H^2 \vee K^2})^2 = H^2 \vee K^2$. □

THEOREM 5. *A group* G *is Hamiltonian if and only if* Δ *is a neutral element of the lattice* $C_w(G)$.

<u>Proof</u>.

By 3) and Corollary 2. □

<u>COROLLARY</u> 6. *A group* G *is Hamiltonian if and only if* Δ *is an exceptional element of the lattice* $C_W(G)$.

<u>**Proof.**</u>

By Lemma 4 and Theorem 5. □

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REZIME

O NEKIM MREŽNIM KARAKTERIZACIJAMA HAMILTONOVIH GRUPA

U radu je dokazano da je proizvoljna grupa Hamiltonova ako i samo ako je mreža njenih slabih kongruencija modularna.

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