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THE DIAGONAL RELATION IN THE LATTICE OF WEAK CONGRUENCES AND THE REPRESENTATION OF LATTICES

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ABSTRACT.

The problem of representation of an algebraic lattice L by the lattice $C_w(A)$ of weak congruences of an algebra A (i.e. of all congruences on all the subalgebras of A) is closely related to the localization of a diagonal relation Δ in $C_w(A)$. Δ is a co-distributive element in $C_w(A)$ ([6]), and it can also be neutral ([6]) and exceptional (as shown here).

Here we shall discuss the following question:

If a is a co-distributive (neutral, exceptional) element of an algebraic lattice L, is there an algebra A, such that $C_w(A) \simeq L$, and that $f(a)=\Delta$ under that isomorphism?

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Every positive answer to this question gives a representation of L by $C_w(A)$.

We shall show that the answer is positive if a is a minimal element of L. Thus, every algebraic lattice has at least one respresentation by weak congruences. This representation turns out to be the only one for lattices having no co-distributive elements different from 1 and 0.

However, the general answer to the above mentioned question is negative, because of some algebraic properties of $C_w(A)$ that we prove here. In order to give a lattice-theoretic interpretation of these algebraic results, we define Δ-suitable element of an algebraic lattice as the one that satisfies all the mentioned necessary conditions under which an element corresponds to Δ.

1. INTRODUCTION.

A *weak congruence* ρ on an algebra $\mathcal{A}=(A,F)$ is a symmetric, transitive and compatible relation on A, which is also weakly reflexive: if c is a constant on $\mathcal{A}$, then $c\rho c$ ([4]). We denote by $C_w(\mathcal{A})$, $C(\mathcal{A})$ and $S(\mathcal{A})$ the algebraic lattices of weak congruences, of (ordinary) congruences, and of subalgebras of $\mathcal{A}$, respectively.

$C_w(\mathcal{A})$ coincides with the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ (under set inclusion). Moreover, $C(\mathcal{A})$ is its sublattice (as a filter generated by $\Delta=\{(x,x)\mid x\in A\}$), and $S(\mathcal{A})$ is a retract of $C_w(\mathcal{A})$ (ideal generated by Δ) ([4]). Subalgebras are represented in $C_w(\mathcal{A})$ by diagonal relations $\mathcal{B}\in S(\mathcal{A})$ which correspond to $d_{B^2}=\{(x,x)\mid x\in B\})$.

We shall use the following special elements of a bounded lattice L: if $a\in L$ then

(I) a is *co-distributive*, if for all $x,y\in L$

$$a\wedge(x\vee y)=(a\wedge x)\vee(a\wedge y),$$

or, equivalently, if $m_a: x\mapsto x\wedge a$ is a homomorphism ([1]);

thus, a co-distributive element uniquely determines the following sublattices of L: $(a]$, $[a)$ (ideal and filter generated by a), and for every $x \in L$ $[x]_{\theta_a}$, where θ_a is a congruence on L induced by m_a;

(II) a is *neutral*, if $m_a: x \to x \wedge a$, and $n_a: x \to x \vee a$ are homomorphisms, and $f_a: x \to (x \wedge a, x \vee a)$ is an embedding of L into $(a] \times [a)$ ([1]);

(III) a is *exceptional*, if it is neutral and the classes of the congruence induced by m_a have maximum elements which form a sublattice M_a of L ([3]).

For $x \in L$, let $\bar{x}$ be the maximum of the class to which x belongs; thus $m_a(x) = m_a(\bar{x})$.

Considering the algebraic lattice $C_w(\mathcal{A})$ of an algebra $\mathcal{A}$, we get the following statements, the proofs of which are straightforward.

1. Δ is a co-distributive element of $C_w(\mathcal{A})$.

2. The mapping $m_\Delta: \rho \to \rho \wedge \Delta$ $(\rho \in C_w(\mathcal{A}))$ is a lattice homomorphisam (the homomorphic image is $S(\mathcal{A})$, as already pointed out).

3. The mapping $n_\Delta: \rho \to \rho \vee \Delta$ $(\rho \in C_w(\mathcal{A}))$ is a lattice homomorphism if and only if $\mathcal{A}$ satisfies the Congruence Intersection Property (CIP) ([4]). In that case, Δ is a distributive elements of $C_w(\mathcal{A})$.

4. Δ is neutral in $C_w(\mathcal{A})$ if and only if $\mathcal{A}$ satisfies both CIP and CEP (the Congruence Extension Property) ([6]).

5. The classes of the congruence induced by m_Δ have maximum elements forming a set $M_\Delta = \{B^2 \mid B \in S(A)\}$ (M_Δ is not necessarily a sublattice of $C_w(A)$).

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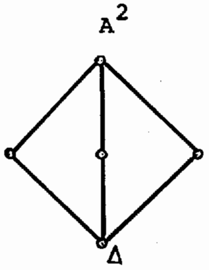
Now, let L be an arbitrary algebraic lattice. In order to give an answer to the question at the beginning of the article, we shall start with the fact that the minimal element, 0, is co-distributive in L. Thus, we have the following Represetation Theorem.

Theorem 1. *Let* L *be an algebraic lattice. Then there is an algebra* A *such that its lattice of weak congruence* $C_w(A)$ *is isomorphic to* L.

Proof. If L is algebraic, then there is an algebra $A_1 = (A, F_1)$ such that its lattice of (ordinary) congruences $C(A_1)$ is isomorphic to L (G.Grätzer, E.T.Schmidt, [2]. Now, let $A = (A, F)$ be the algebra constructed on the same set A as A_1, and with the same set of non-nullary operations, but in which the set of constants K is the whole set A (K=A), i.e. such that $F = A \cup F_1$. Then, obviously, $C_w(A) = C(A) = C(A_1)$, and we are done. □

If L has no co-distributive element other than 0 and 1, then the only way to resperesent it by weak congruences is the one given in Theorem 1.

EXAMPLE 1. The lattice M_3(Fig.1) can be represented by the latice of weak congruences only by adding to the set of operations all the elements of a suitable algebra (Klein's group, for example) as constants.



$A=(\{e,a,b,ab\},\cdot,^{-1},e,a,b,ab)$

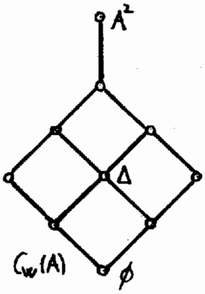
	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

$C_w(A)=C(A)$

Fig. 1

In the following two examples Δ is a neutral and exceptional element (respectively) in the lattice of weak congruences of the corresponding algebra. An example in which Δ is co-distributive (but not neutral) is $C_w(S_3)$, where S_3 is a symmetric group on a three-element set (see [5]).

Δ-neutral, but not exceptional

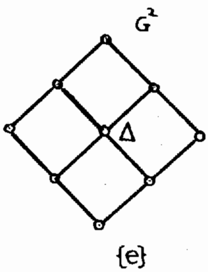


$A=(A,f_1,f_2)$ $A=\{a,b,c,d\}$

A	a	b	c	d
f_1	a	a	d	c

A	a	b	c	d
f_2	b	a	d	c

Fig. 2



G - any cyclic group of order p·q p,q-prim, p≠q.

Δ- exceptional

C_wG.

Fig. 3

However, the localization of an arbitrary co-distributive (neutral, exceptional) element a in an algebraic lattice L is not sufficient for the existence of an isomorphism f between L and the lattice of weak congruences of an algebra (with $f(a)=\Delta$). In order to prove that there are co-distributive elements which could not correspond to Δ, we advance the following propositions.

LEMMA 2. *Let* $\rho_1,\rho_2\in C_w(A),\rho_1\neq\emptyset$[1)] *for an arbitrary algebra* A, *and let* θ_Δ *be the congruence on* $C_w(A)$ *induced by* m_Δ *(see 2.). Then,* $[\rho_1]_{\theta_\Delta}\prec[\rho_2]_{\theta_\Delta}$ *implies that* $|[\rho_2]_{\theta_\Delta}|\neq 1$.

(In other words, there are no pairs of one-element comparable classes modulo θ_Δ, unless the first contains the empty relation.)

<u>Proof.</u> Obvious, since a one-element class consists of a diagonal relation of a one-element (or empty) subalgebra. □

An immediate consequence of this lemma is that in a bounded chain the only element other than 0 which could correspond to Δ is the atom (if there is one). All the other elements in a chain are co-distributive, but the correspoding classes of weak congruences are one-element and

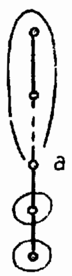


Fig. 4

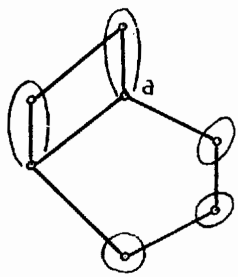


Fig. 5

1) It is possible that the weak congruence is an empty relation (see[4]).

comparable (see Fig. 4 and 5 for the lattices in which co-distributive elements violate Lemma 2).

LEMMA 3. *If* $B,C\in S(A)$ *for an algebra* A, *then* $B\wedge C\neq\emptyset$ *implies* $B^2\vee C^2=(B\vee C)^2$ *in* $C_w(A)$.

Proof. Let $B\wedge C=D\neq\emptyset$. Then

$B^2\vee C^2,(B\vee C)^2\in C(B\vee C)$ (the lattice of ordinary congruences on $B\vee C$). Now, from

$$[D^2]_{B^2\vee C^2}\in S(B\vee C)$$

($[D^2]_{B^2\vee C^2}$ is a class in the congruence $B^2\vee C^2$ to which D^2 belongs), it follows that

$$([D^2]_{B^2\vee C^2})^2\leq B^2\vee C^2 \quad . \tag{1}$$

Clearly,

$$B,C\leq[D^2]_{B^2\vee C^2} \quad . \tag{2}$$

From (1) and (2) we get

$$(B\vee C)^2\leq([D^2]_{B^2\vee C^2})^2\leq B^2\vee C^2 \quad ,$$

i.e. $$(B\vee C)^2\leq B^2\vee C^2 \quad .$$

Since obviously

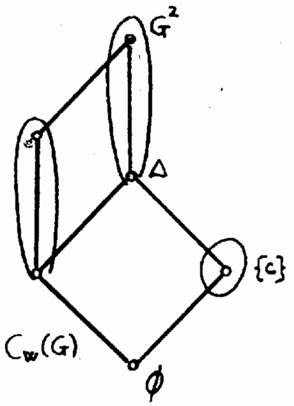
$$B^2\vee C^2\leq(B\vee C)^2 \quad ,$$

finally we have

$$B^2\vee C^2=(B\vee C)^2 \quad . \ \square$$

COROLLARY 4. *If* $\cap(B\,|\,B\in S(A))\neq\emptyset$, *then* M_Δ *is a sublattice of* $C_w(A)$. □

REMARK. The converse of Corallary 4 is not valid: see the groupoid on Fig. 6. M_Δ is there a sublattice of $C_w(G)$, but the minimal element of this lattice is the empty set.



$(G,\cdot)$ $G=\{a,b,c\}$

	a	b	c
a	b	b	c
b	b	a	a
c	a	a	c

Fig. 6

COROLLARY 5. *If* Δ *is neutral element in* $C_w(A)$, *and the minimal subalgebra of* A *is not empty, then* Δ *is exceptional in* $C_w(A)$.

Proof. By Corollary 4 and by (III). □

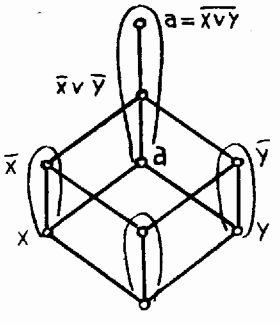


Fig. 7

Lemma 3, and Corollaries 4 and 5 exclude a wide class of co-distributive elements as possible diagonal relations in the representation of lattices by weak congruences. A co-distributive element a in the lattice in Fig. 7 violates Corollary 5, since it is neutral, the minimal class of the congruence induced by m_a has two elements (the corresponding minimal subalgebra could not be void), but a is not exceptional.

For an arbitrary algebra $\mathcal{A}$, and $\mathcal{B}\in S(\mathcal{A})$, $\theta\in C(\mathcal{B})$, let[1])

$$B[\theta] \overset{def}{=} \{a\in A \mid (\exists b\in B)\, a\theta b\}.$$

LEMMA 6. a) $\mathcal{B}\le \mathcal{B}[\theta]\le \mathcal{A}$;

b) *if* $\{\theta_i, i\in I\}\subseteq C(\mathcal{A})$, *and* $B[\theta_i]=B$ *for every* i, *then* $B[\bigvee_{i\in I}\theta_i]=B$.

<u>Proof</u>.

a) Obvious.

b) If $a\in B[\vee\theta_i]$, then there are $a=a_1,\dots,a_{n-1}\in A$, $a_n=b\in B$, and $\theta_{i_1},\dots,\theta_{i_{n-1}}\in\{\theta_i, i\in I\}$, such that $a_j\theta_{i_j}a_{j+1}$. If $a\notin B$, then $a_j\notin B$ for some $j\in\{1,\dots,n-1\}$. But this contradicts the assumption $B[\theta_{i_j}]=B$, since $a_j\theta_{i_j}a_{j+1}$. □

For $\mathcal{B}\in C(\mathcal{A})$, let

$$\theta_B \overset{def}{=} \bigvee(\theta\in C(\mathcal{A}) \mid B[\theta]=B).$$

Obviously, $\theta_B\in C(\mathcal{A})$, and for every $\theta\in C(\mathcal{A})$,

a) $B[\theta]=B \Leftrightarrow \theta\le\theta_B$.

Moreover,

b) $B[\theta_B]=B$.

1) The consideration that follows, concluding with Theorem 9, was suggested by E. Kiss.

(θ_B is thus the greatest congruence on A in which the subalgebra B is a union of some classes.)

LEMMA 7. *For every* $B\in S(A), \theta\in C(A)$

$$B^2\vee\theta = B[\theta]^2\vee\theta, \text{ in } C_W(A)$$

Proof. By the definition of $B[\theta]$,

$$B[\theta]^2 \leq \cap\,(\rho\in C(A) \mid B^2\vee\theta\leq\rho) = B^2\vee\theta.$$

Thereby, $B[\theta]^2\vee\theta\leq B^2\vee\theta$.

The opposite inequality is trivial. □

COROLLARY 8. *If* $B[\theta]=A$, *then* $B^2\vee\theta=A^2$. □

THEOREM 9. *For an algebra* A, *let* $B\in S(A)$, *and* $A\succ B$ *(*B *is a coatom in* $S(A)$ *Then,*

$$\vee(\theta\in C(A) \mid \theta\vee B^2 < A^2) \neq A^2.$$

Proof. For $B\in S(A)$, $A\succ B$, and $\theta\in C(A)$ $\theta\vee B^2<A^2$ implies $B[\theta]\neq A$ (by Corollary 8). Hence, $B[\theta]=B$ (since, by assumption, there is no subalgebra between A and B). Thus, $\theta_B\geq\theta$, and $A^2>\theta_B\geq\vee(\theta\in C(A) \mid \theta\vee B^2<A^2)$. □

So far there is no name for an element which satisfies all the listed necessary conditions under which it corresponds to a diagonal relation in a representation of a lattice by weak congruences. Therefore, we shall give the following definition (for the notation, see the introduction).

DEFINITION 1. An element a of an algebraic lattice L is Δ-*suitable* if it is co-distributive and satisfies the following conditions:

i) $x\wedge y\neq 0$ implies $\overline{x}\vee\overline{y}=\overline{x\vee y}$, for all $x,y\in L$;

ii) if $x\neq 0$ and $\overline{x}<y$, then $\overline{y\wedge a}\neq y\wedge a$, for all $x,y\in L$;

iii) if $x\neq 0$ and $x=\wedge(y\,|\,y\in L\setminus\{0\})$, then $x\neq\overline{x}$, for every $x\in L$;

iv) if $x\in L$ and $\overline{x}\prec 1$ ($\overline{x}$ is covered by 1), then $\vee(y\in L\,|\,y\geq a, y\vee\overline{x}<1)\neq 1$.

It is clear that a Δ-suitable element must be co-distributive. Condition i) is required by Lemma 3, and ii) as well as iii) by Lemma 2; iv) follows by Theorem 9.

Obviously, a Δ-suitable element which is neutral, must be exceptional, if $0\neq\overline{0}$.

As a final example, we note that in a free distributive lattice with three generators ($F_D(3)$, see Fig.8), these generators are the only Δ-suitable elements.

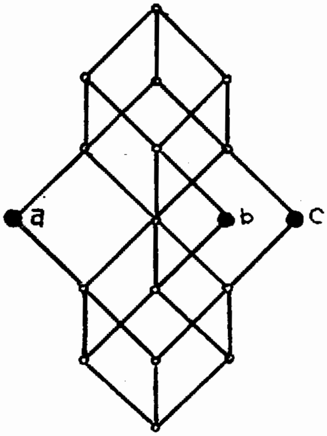


Fig. 8

Since a Δ-suitable element satisfies only some necessary conditions under which it corresponds to a diagonal relation, it is still an open problem whether these conditions are sufficient,as well. To be precise:

PROBLEM. If a is a Δ-suitable element of an algebraic lattice L, is there an algebra A, such that $C_W(A)\cong L$, and that $f(a)=\Delta$ under that isomorphism?

REFERENCES

[1] G.Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin, 1978.

[2] G.Grätzer, E.T.Schmidt, *Characterizations of congruence lattices of abstract algebras*, Acta Sci.Math. (Szeged), 24 (1963), 34-59.

[3] N.R.Reilley, *Representations of lattices via neutral elements*, Algebra Universalis, 19(1984), 341-354.

[4] G.Vojvodić, B.Šešelja, *On the Lattice of Weak Congruence Relations*, Algebra Universalis 25(1988), 121-130.

[5] G.Vojvodić, B.Šešelja, *A Note on the Modularity of the Lattice of Weak Congruences on the Finite Group*, Contributions to General Algebra 5, Proceedings of the Salzburg Conference, May, 29-June 1, 1986, 415-419.

[6] G.Vojvodić, B.Šešelja, *On CEP and CIP in the Lattice of Weak Congruences*, Proceedings of the Conference Algebra and Logic, Cetinje, Yugoslavia, 1986, 221-227.

REZIME

DIJAGONALNA RELACIJA U MREŽI SLABIH KONGRUENCIJA I REPREZENTACIJA MREŽA

U radu je dokazana teorema reprezentacije proizvoljne algebarske mreže uz pomoć mreže slabih kongruencija algebre. S obzirom da to predstavljanje zavisi od lokalizovanja elementa mreže koji se u reprezentaciji preslikava u dijagonalnu relaciju, daju se neki potrebni uslovi pod kojima proizvoljan kodistributivan element algebarske mreže odgovara dijagonali. Definiše se odgovarajići Δ-podoban elemenat proizvoljne algebarske mreže.

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