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FIXED POINT THEOREMS IN TAKAHASHI CONVEX METRIC SPACES

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ABSTRACT

In this paper a generalization of a fixed point theorem from [7] is proved for a class of Takahashi convex metric spaces.

1. INTRODUCTION

In 1970 Takahashi [6] introduced the definition of convexity in a metric space and generalized some important fixed point theorems previously proved for Banach spaces. Subsequently, Machado [4], Talman [7], Gauy and Singh [1] , Hadžić and Gajić [2], Gajić [3], among others have obtained additional results in this setting. This paper is a continuation of these investigations.

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2. PRELIMINARIES

Definition 1. *Let* X *be a metric space and* I *be the closed unit interval. A mapping* $W: X\times X\times I\to X$ *is said to be a convex structure on* X *iff for all* $x,y\in X$, $\lambda\in I$,

$$d(u,W(x,y,\lambda))\leqslant\lambda d(u,x)+(1-\lambda)d(u,y),\ \textit{for all}\ u\in X.$$

X *together with a convex structure is called a Takahashi convex metric space.*

Any convex subset of a Banach space is a Takahashi convex space with $W(x,y,\lambda)=\lambda x+(1-\lambda)y$.

Definition 2. *Let* X *be a convex metric space. A nonempty subset* K *of* X *is convex iff* $W(x,y,\lambda)\in K$ *whenever,* $x,y\in K$ *and* $\lambda\in I$.

Takahashi has shown that open and closed balls are convex and that the arbitrary intersection of convex sets is convex ([5]).

For arbitrary $C\subset X$ *let:*

$$\tilde{W}(C):=\{W(x,y,\lambda):x,y\in C,\lambda\in[0,1]\}. \qquad (1)$$

It is easy to see that $\tilde{W}:P(X)\to P(X)$ *is a mapping with properties*

1. $C\subseteq\tilde{W}(C)$, *for* $C\in P(X)$;
2. $C\subseteq B$ *implies* $\tilde{W}(C)\subseteq\tilde{W}(B)$, *for* $B,C\in P(X)$;
3. $\tilde{W}(C\cap B)\subseteq\tilde{W}(C)\cap\tilde{W}(B)$, *for any* $B,C\in P(X)$.

Using this notation we can say that $K\subseteq X$ *is convex iff* $\tilde{W}(K)\subseteq K$.

A few additional definitions and propositions will be needed subsequently.

Definition 3. *A convex metric space* X *is said to have* Property *(C) iff every decreasing net of nonempty, closed convex subsets of* X *has a nonempty intersection.*

Remark. Every weakly compact convex subset of a Banach space has Property (C).

Definition 4. *Let* X *be a convex metric space and* A *a nonempty closed convex bounded set in* X. *For* $x \in X$ *we set:*

$$r_x(A) = \sup_{y \in A} d(x,y),$$

$$r_x(A) = \inf_{x \in A} r_x(A),$$

$$A_c = \{x \in A : r_x(A) = r(A)\}, \quad (\textit{centre of } A);$$

$$\delta(A) = \sup\{d(x,y) : x,y \in A\}; \quad (\textit{diametar of } A).$$

Definition 5. *A point* $x \in A$ *is a* diametral point *of* A *iff* $\sup_{y \in A} d(x,y) = \delta(A)$.

Definition 6. *A convex metric space is said to have a* normal structure *iff for each closed bounded convex subset* A *of* X, *which contains at least two points, there exists* $x \in A$ *which is not a diametral point for* A.

Remark. Any compact convex metric space has a normal structure ([1]).

For sets $K, H \subset X$ by $\partial_F K$ we shall denote the boundary of K relative to H. If K is closed then:

$$\partial_H K=\{z\in K \mid B(z,r)\cap (H\setminus K)\neq \emptyset \text{ for each } r>0\}$$

$(B(z,r)=\{x\in X \mid d(x,z)<r\})$.

Lemma 1. [1] *Let* H *and* K *be two closed subset of the Takahashi convex metric space* X *such that* $H\cap K\neq\emptyset$

If $W: X\times X\times I\to X$ *is continuous in* $\lambda\in I$ *and* H *is convex, then* $\partial_H K=\emptyset \Leftrightarrow H\subseteq K$.

Lemma 2. *The continuous image of the convex subset* K *of a Takahashi convex metric space with a continuous convex structure* W *is a connected set.*

Proof. Since K is convex and W is continuous, K is pathwise connected, and since a continuous image of a pathwise connected set is pathwise connected, it is connected too.

Now, let us recall that the convex hull of a set A, $A\subset X$ is the intersection of all the convex sets in X containing A, and it is denoted by *conv* A.

It is obvious that if A is a convex subset of a convex metric space X, then

$$\tilde{W}^n(A)=\tilde{W}(W(\dots\tilde{W}(A))\dots)\subseteq A, \text{ for any } n\in N.$$

For $n\in N$ we set

$$A_n=\tilde{W}^n(A).$$

The sequence $\{A_n\}_{n\in N}$ is increasing so *lim inf* and *lim sup* exist and

$$\limsup A_n=\liminf A_n=\lim A_n=\bigcup_{n\in N} A_n.$$

Proposition 1. [3] *Let X be a Takahashi convex metric space.*
Then,

$$conv\ A = lim\ A_n = \bigcup_{n\in N} A_n\ , \quad (A \subset X). \qquad (2)$$

In the remainder of the paper, (X,d) will denote a complete metric space with a convex structure W.

Proposition 2. [3] *For any subset A of (X,d)*

$$\delta(conv\ A) = \delta(A).$$

Lemma 3. *Let* $K \subset H$ *be two nonempty closed subsets in (X,d), let H be convex and let W be continuous in* λ*. If* $x \in K$ *and* $y \in H \setminus K$ *then there exists* $\lambda_o \in (0,1]$*, such that* $W(x,y,\lambda_o) \in \partial_H K$.

Proof. Let $L = \{\lambda \in [0,1] \mid W(x,y,\lambda) \in K\}$ since L is bounded and nonempty ($1 \in L$), there exists *inf* L. Let $\lambda_o =$ $=$ *inf* L. If $\lambda_o \in (0,1]$, then for every $n \in N$ there exist λ_n, $\lambda_o \leq \lambda_n \leq 1$, $\lambda_n \to \lambda_o$ and so that $W(x,y,\lambda_n) \in K$. Since W is continuous $W(x,y,\lambda_o) \in K$, too. On the another hand, there exist $\{\tilde{\lambda}_n\} 0 \leq \tilde{\lambda}_n < \lambda_o, \tilde{\lambda}_n \to \lambda_o$ so that $W(x,y,\tilde{\lambda}_n) \in H \setminus K$ and, then, $W(x,y,\tilde{\lambda}_n) \to W(x,y,\lambda_o)$ so we prove that $W(x,y,\lambda_o) \in \partial_H K$.

For $\lambda_o = 0$, since $d(y, W(x,y,\lambda)) = \lambda d(x,y)$, one can prove that $y \in \bar{K} = K$. Contradiction!

3. RESULTS

Theorem 1. *Let (X,d) be with a continuous convex structure and let $H,K(K \subseteq H)$ be nonempty closed convex subsets of (X,d). Further, let H be a normal subset and K a bounded set with Property (C).*

If $A:K \to H, A(\partial_H K) \subset K$, and A is a mapping which, for all $x,y \in K$ satisfies the inequality:

$$d(Ax,Ay) \leq ad(x,y)+b[d(x,Ax)+d(y,Ay)]+c[d(x,Ay) + d(y,Ax)], \tag{3}$$

where a,b,c are nonnegative constants such that $a+2b+2c \leq 1, a+b>0$, then A has a fixed point.

Poof. We shall assume that $a+2b+2c=1$. For fixed $\tilde{x}_o \in K$ and arbitrary $x \in K$ we have from (3):

$$d(Ax,A\tilde{x}_o) \leq ad(x,\tilde{x}_o)+b[d(x,Ax)+d(\tilde{x}_o,A\tilde{x}_o)] + c[d(x,A\tilde{x}_o)+d(\tilde{x}_o,Ax)] \leq ad(x,\tilde{x}_o)+bd(x,\tilde{x}_o) + bd(\tilde{x}_o,A\tilde{x}_o)+bd(A\tilde{x}_o,Ax)+bd(\tilde{x}_o,A\tilde{x}_o)+cd(x,\tilde{x}_o) + cd(A\tilde{x}_o,\tilde{x}_o)+cd(\tilde{x}_o,A\tilde{x}_o)+cd(A\tilde{x}_o,Ax)$$

and further,

$$d(Ax,A\tilde{x}_o) \leq \frac{a+b+c}{1-b-c} d(x,\tilde{x}_o) + \frac{2(b+c)}{1-b-c} d(\tilde{x}_o,A\tilde{x}_o).$$

This and the boundness of K means that A is bounden on K.

Let, as in [7], $\mathcal{F}$ be the family of all the closed convex subsets of H so that for $F \in \mathcal{F}$, $F \cap K \neq \emptyset$ and

$A: F \cap K \to F$. Since $H \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Let $\{F_\alpha\}$ be a decreasing chain of sets of $\mathcal{F}$ and let $F_0 = \cap F_\alpha$. Note that $F_0 \cap K$ is nonempty since $\{F_\alpha \cap K\}$ is a decreasing chain of a nonempty closed convex subset of a set with Property (C). Also, since $A: F_\alpha \cap K \to F_\alpha$, for each α, clearly $A: F_0 \cap K \to F_0$. Since F_0 is a closed convex, $F_0 \in \mathcal{F}$ so it follows by Zorn's Lemma that $\mathcal{F}$ has a minimal element.

Let F be a minimal element and suppose $\partial_F K \neq \emptyset$. We shall prove that $M \overset{\text{def}}{=\!=\!=} F \cap K$ has only one element. Suppose that M has more than one element. Then, center M_c is a nonempty closed convex set so that:

$$\delta(M_c) \leq r(M) < \delta(M).$$

Furthermore, if $\overline{conv}\, N$ denote the closed convex hull of set N, then we have

$$(\overline{conv} A(M)) \cap K \supseteq A(M) \cap K \supseteq A(\partial_F K) \cap K = A(\partial_F K) \neq \emptyset$$

and

$$A(\overline{conv} A(M) \cap K) \subseteq A(F \cap K) = A(M) \subset \overline{conv} A(M).$$

From this two facts and from the minimality of the set F we get

$$\overline{conv} A(M) = F.$$

This and the boundedness of A give $\delta(F) < +\infty$.

Let $y \in M_c$. If $Ay \in M$ set $x_0 = y$. If $Ay \notin M$, then $y \notin$

$\notin \partial_F K$ and by Lemma 3 there exists $0<\lambda_o<1$ such that $W(y,Ay,\lambda_o) \in \partial_F K$. In this case put $x_o=W(y,Ay,\lambda_o)$. In any case, we have $x_o, Ax_o \in M$ (since $A:\partial_F K \to K$) and that

$$r_0 \stackrel{\text{def}}{\equiv} \sup \{d(x_0,z) \mid z\in M\} < \delta(F).$$

Further, for all $x\in M$ we have:

$$d(Ax,Ax_o) \leq ad(x,x_o)+bd(x_o,Ax_o)+bd(x,Ax) +$$
$$+ cd(x_o,Ax)+cd(x,Ax_o) \leq (a+b)r_o+(b+2c)\delta(F),$$

so, since we prove that $F = \overline{conv}A(M)$,

$$r \stackrel{\text{def}}{\equiv} \sup \{d(Ax_o,z) \mid z\in F\} < \delta(F). \tag{4}$$

Now, we shall define a transfinite sequence of the set $\{M_\alpha\}$ setting

$$M_0 = \{Ax_0\}$$

$$M_\alpha = \overline{conv}((M_{\alpha-1} \cap M) \cup A(M_{\alpha-1} \cap M)) \text{ if } \alpha-1 \text{ exists,}$$

$$M_\alpha = \overline{\bigcup_{\beta<\alpha} M_\beta} \qquad \text{if } \alpha-1 \text{ does not exists.}$$

Obviously, the sets M_α are nonempty convex (W is continuous) and closed subsets of the set F for which $M_\alpha \cap K \neq \emptyset$. For $\alpha<\alpha'$ it is $M_\alpha \subset M_{\alpha'}$. Taking an ordinal number α^* greater than the cardinal number of the power set of F, we see that in the sequence $\{M_\alpha \mid 0\leq\alpha\leq\alpha^*\}$ there must be repetitions. If α_o is an ordinal number for which $M_{\alpha_0+1}=M_{\alpha_o}$ $(0\leq\alpha_o<\alpha^*)$, then we have

$$A(M_{\alpha_o} \cap M) \subseteq \overline{conv}((M_{\alpha_o} \cap M) \cup A(M_{\alpha_o} \cap M)) =$$

$$= M_{\alpha_o+1} = M_{\alpha_o}. \tag{5}$$

We shall prove for all $(0 \leq \alpha \leq \alpha_o)$ that

$$\delta(M_\alpha) \leq r. \tag{6}$$

For $\alpha=0$, (6) is true. Let $0<\alpha\leq\alpha_o$ and suppose that for all β, $0\leq\beta<\alpha$

$$\delta(M_\beta) \leq r \tag{7}$$

and

$$d(x,Ay) \leq r \quad \text{for} \quad x,y \in M_\beta \cap M. \tag{8}$$

Suppose that $\alpha-1$ exists. Taking a sequence $\{\varepsilon_n\}, \varepsilon_n>0, \varepsilon_n\to 0$ we can find $\tilde{x}_n,\tilde{y}_n \in M_\alpha$, so that

$$\delta(M_\alpha) - \varepsilon_n \leq d(\tilde{x}_n,\tilde{y}_n), \quad n=1,2,\ldots \ . \tag{9}$$

Further, we may assume that one of the following is true:

i) $\tilde{x}_n,\tilde{y}_n \in M_{\alpha-1} \cap M$, $n=1,2,\ldots$;

ii) $\tilde{x}_n=Ax_n, \tilde{y}_n=Ay_n, x_n,y_n \in M_{\alpha-1} \cap M$, $n=1,2,\ldots$;

iii) $\tilde{x}_n \in M_{\alpha-1} \cap M, \tilde{y}_n=Ay_n, y_n \in M_{\alpha-1} \cap M$, $n=1,2,\ldots$.

If i) or iii) is true, then from (7), (8), (9) it follows immediately that $\delta(M_\alpha) \leq r$. If ii) is true then (3), (7), (8) and (9) give $\delta(M_\alpha) \leq r$, so inequality (7) is proved for $\alpha=\beta$.

Let $x \in M_\alpha \cap M$. Then for a $x \in M_\alpha \cap M$ and a given $\varepsilon>0$ we may take $x' \in$ *conv* $\{(M_{\alpha-1} \cap M) \cup A(M_{\alpha-1} \cap M)\}$ for which

$$d(x,Ax) < \varepsilon + d(x',Ax).$$

Using Proposition 1 we have that there exists $k_o \in N$, so that $x' \in \tilde{W}^{k_o}((M_{\alpha-1} \cap M) \cup A(M_{\alpha-1} \cap M))$. By (3), (7), (8) we get as in [7] that:

$$d(x,Ax) < \varepsilon + \sum_{i \in I_1} \omega_i d(u_i,Ax) + (a+b+c)(1-\omega)r +$$

$$+b(1-\omega)d(x,Ax) + c \sum_{i \in I_2} \omega_i d(u_i,Ax) \tag{10}$$

$I_1 \cup I_2 = I$, card $I \leq 2^{k_o}$, $\omega_i \geq 0$, $i \in I$, $\sum_{i \in I} \omega_i = 1$

$\omega = \sum_{i \in I_1} \omega_i$ and $u_i \in M_{\alpha-1} \cap M$, for all $i \in I$.

This means that the set S of all β, $0 \leq \beta < \alpha$ for which

$$d(x,Ax) \leq \varepsilon + \sum_{i \in I} \gamma_i d(v_i,Ax) + \gamma(a+b+c)r +$$

$$+\gamma b d(x,Ax) \tag{11}$$

for some $v_i \in M_\beta \cap M$ and some real numbers $\gamma, \gamma_i \geq 0$ (I-finit set) for which

Now, in a similar way, using (12) one can prove that (8) is true for $\beta=\alpha$.

In the case that $\alpha-1$ does not exist, from (7) for $\beta<\alpha$ it follows immediately that (7) holds for $\beta=\alpha$. Inequality (8), for $\beta=\alpha$, may be proved in a similar way as in the case when $\alpha-1$ exists, so we have that (7) and (8) are valid for all α, $0\leq\alpha\leq\alpha_0$.

Finally, we have that M_{α_0} is a nonempty closed convex subset of the set F with properties that

a) $\delta(M_{\alpha_0})\leq r<\delta(F)$

b) $M_{\alpha_0}\cap K=M_{\alpha_0}\cap M\neq\emptyset$

c) $A(M_{\alpha_0}\cap K)\subset M_{\alpha_0}$.

But this is a contradiction to the minimality of the set F, therefore $M=F\cap K$ has only one point.

Let $\{x^*\}=F\cap K$. From $\partial_F K\neq\emptyset$. We have $\{x^*\}=\partial_F K$. Then, $Ax^*\in F\cap K=x^*$ and x^* is the fixed point of A.

If $\partial_F K=\emptyset$ then, by Lemma 1, $F\subset K$ and $A:F\cap K\to F$ would imply $A:F\to F$. If F has more than one point, from the fact that H has a normal structure, one can see that there exist $x_0\in F$, for which

$$\sup\{d(x_0,y)\,|\,y\in F\}<\delta(F).$$

We can now construct the sequence $\{M_\alpha\}$, with $M_0=\{x_0\}$ and repeating the above procedure, with some simplifications, we again get a contradiction. Hence, F has only one element. This and $A:F\to F$ imply that A has a fixed point.

Theorem 2. *Let* (X,d) *be with a continuous convex structure,* H *and* K, $K \subset H$, *nonempty closed convex subset of* X. *Further, let* H *be a bounded normal subset and* K *with Property (C). If* $A:K\to H, A(\partial_H K)\subset K$ *and if* A *satisfies (3), where* $a,b,c\geq 0$ *and* $a+2b+2c\leq 1$, *then* A *has a fixed point.*

Proof. Let $\mathcal{F}$, F, M have the same meaning as in the proof of Theorem 1.

Suppose that $M=F\cap K$ has more than one element and $\partial_F K\neq \emptyset$. Since $F\subset H$ and H has a normal structure, there exists $x_0'\in F$ for which

$$r_0'= sup\{d(x_0', y):y\in F\}<\delta(F).$$

Define $x_0\in M$ in the following way :

1) if $x_0'\in M$ put $x_0=x_0'$

2) if $x_0'\notin M$ take $x_0''\in M\setminus\partial_F M$, chose $\lambda\in(0,1)$ such that $W(x_0'',x_0',\lambda)\in\partial_F M=\partial_F K$ and $x_0=W(x_0'',x_0',\lambda)$

(Note that in the case $x_0'\notin M$ we have that $M\setminus\partial_F M\neq\emptyset$ because in the opposite case $\partial_F M=M$ and therefore $A:M\to M$. From the minimality of the set F, we now have M=F so $x_0'\in M$).
It is easy to see that in any case we have

$$sup\{d(x_0,y)\,|\,x\in F\}< \delta(F).$$

Now, we can construct the sequence $\{M_\alpha\}$, starting with $M_0=\{x_0\}$ and so on.

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REZIME

TEOREME O NEPOKRETNOJ TAČKI U TAKAHAŠIJEVIM KONVEKSNIM METRIČKIM PROSTORIMA

U ovom radu dokazane su teoreme o nepokretnoj tački u klasi Takahašijevih konveksnih metričkih prostora koje uopštavaju rezultate za normirane prostore iz rada [7].

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