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SOME PROPERTIES OF SUBSETS AND ALMOST CLOSED MAPPINGS

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ABSTRACT

In the paper some properties of α-Hausdorff subsets and almost closed mappings are studied.

1. INTRODUCTION

No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is *regularly open* iff IntClA=A. A subset A of a space X is *regularly closed* iff ClIntA = A, [9].

A subset A of a space X is α- *paracompact (α-nearly paracompact)* iff for every open (regularly open) cover

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$\mathcal{U}$ of A there is an open X-locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ and covers A, [13],[5].

A subset A of a space X is α-paracompact (*α-nearly paracompact) with respect to a subset* B iff for every open (regularly open) cover $\mathcal{U}=\{U_i : i\in I\}$ of A there exists an open family $\mathcal{V}=\{V_j : j\in J\}$ such that:

- $\mathcal{V}$ refines $\mathcal{U}$,
- $A \subset \cup\{V_j : j\in J\}$,
- $\mathcal{V}$ is locally finite at each point $x\in B$, [6].

Subsets A and B of a space X are *mutually α-paracompact (mutually α-nearly paracompact)* iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and the subset B is α-paracompact (α-nearly paracompact) with respect to the subset A, [6].

A subset A of a space X is *α-nearly compact or N-closed* iff every regularly open cover $\mathcal{U}=\{U_i : i\in I\}$ of A has a finite subcover of A, [1].

A space X is *nearly compact* iff every regularly open cover of has a finite subcover, [11].

A space X is *locally nearly compact* iff for each point $x\in X$, there exists an open neighbourhood U of x such that ClU is α-nearly compact, [1].

A subset A of a space X is *α-Hausdorff* iff for any two points a,b of a space X, where $a\in A$ and $b\in X\setminus A$, there are disjoint open sets U and V containing a and b respectively. A subset A of a space X is *α-regular (α-almost regular)* iff for any point $a\in A$ and any open (regularly open) subset U containing a there is an open subset V such that $a\in V\subset ClV\subset U$,[7];[4].

A space X is *almost regular* iff for any regularly closed set F and any point $x\notin F$, there are disjoint open sets containing F and x respectively, [10].

A mapping $f:X\to Y$ is *almost closed (almost open)* iff for any regularly closed (regularly open) set F of X,f(F) is closed (open) in Y, [9].

A mapping $f:X\to Y$ is *almost continuous* at a point $x\in X$ iff for every open neighbourhood M of f(x) there is an open neighbourhood N of X such that $f(N)\subset \mathrm{IntCl}M$. f is almost continuous iff it is almost continuous at each point of X, [9].

2. RESULTS

The following theorem was proved in [7]:

Theorem A. If A is an α-regular α-paracompact subset of a space X, then ClA is α-paracompact.

We can generalize this result with the following results:

Theorem 2.1. If A is an α-regular α-paracompact subset with respect to a subset B, then ClA is α-paracompact with respect to B.

Proof. It is similar to the proof of Theorem 2.4 in [7].

Theorem 2.2. If A is an α-almost regular α-nearly paracompact subset with respect to a subset B, then ClA is α-nearly paracompact with respect to B.

Proof. It is similar to the proof of Theorem 3.2 in [4].

Lemma 2.1. Let

$$\mathcal{U}=\{U_i:i\in I\}$$

be a family of open α-regular subsets of a space X such that:

a) $\mathcal{U}$ is locally finite at each point of a subset B

b) U_i is α-paracompact with respect to B, for each $i \in I$.

Then, $U = \cup\{U_i : i \in I\}$ is an open α-regular subset which is α-paracompact with respect to B.

Proof. By Lemma 2.1 in [2], the set U is α-regular. Let $\mathcal{V} = \{V_j : j \in J\}$ be an open covering of U. Then, $\{V_j \cap U_i : j \in J\}$ is an open covering of U_i, for each $i \in I$. Since U_i is α-paracompact with respect to B, there is a family $\mathcal{V}_i = \{D_k : k \in K^i\}$ of open sets such that:

- $\mathcal{V}_i$ refines $\{V_j \cap U_i : j \in J\}$,
- $U_i \subset \cup\{D_i : D_i \in \mathcal{V}_i\}$,
- $\mathcal{V}_i$ is locally finite at each point of B.

Consider the family

$$\mathcal{V} = \{D_k : k \in K^i, i \in I\}.$$

It follows that

- $\mathcal{V}$ refines $\mathcal{V}$,
- $U \subset \cup\{D : D \in \mathcal{V}\}$,
- $\mathcal{V}$ is locally finite at each point of B.

Thus, U is α-paracompact with respect to B.

Similarly, we can prove the next result:

Lemma 2.2. Let

$$\mathcal{U} = \{U_i : i \in I\}$$

be a family of regularly open α-almost regular subset of a space X such that:

a) $\mathcal{U}$ is locally finite at each point of a subset B,

b) for each $i\in I$, U_i is α-nearly paracompact with respect to B.

Then, $U=\cup\{U_i:i\in I\}$ is an open α-almost regular subset which is α-nearly paracompact with respect to B.

Theorem 2.3. Let

$$\mathcal{U}=\{U_i:i\in I\}$$

be a family of open α-regular subsets of a space X such that:

a) $\mathcal{U}$ is locally finite at each point of $X\setminus U\neq\emptyset$ $(U=\cup\{U_i:i\in I\})$,

b) U_i is α-paracompact with respect to $X\setminus U$, for each $i\in I$.

Then, U is on open - and - closed α-regular subset which is α-paracompact with respect to $X\setminus U$.

Proof. By Lemma 2.1, $\tilde{U}$ is an open α-regular subset which is α-paracompact with respect to $X\setminus U$. By Theorem 2.6. in [6], it follows that there is an open set V such that

$$U\subset V\subset \mathrm{Cl}V\subset U.$$

Thus $\mathrm{Cl}U=U$. Hence, the result.

In [12], Singal M.K. and Arya S.P. proved the next theorem:

Theorem B. Every nearly paracompact Hausdorff space is almost regular. In that theorem the Hausdorff property can be weakened as is shown by following result:

Theorem 2.4. Let X be a paracompact (nearly paracompact) space such that every closed (regularly closed) set is α-Hausdorff. Then X is regular (almost regular).

Proof. Let X be a paracompact (nearly paracompact) space and let F be any closed (regularly closed) subset of a space X and let $x\notin F$. Since every closed (regularly closed) subset of a paracompact (nearly paracompact) space is α-paracompact (α-nearly paracompact) and F is α-Hausdorff, it follows that there are open (regularly open) sets U and V such that

$$x \in U, F \subset V, U \cap V = \phi.$$

It follows that X is regular (almost regular).
Similarly, we have

Corollary 2.1. Let X be a compact (nearly compact) space such that every closed (regularly closed) subset is α-Hausdorff. Then, X is regular (almost regular).

Theorem 2.5. Let $f: X \to Y$ be a closed almost continuous mapping of a space X onto a locally compact space Y such that for each $y \in Y$ $f^{-1}(y)$ is α-Hausdorff α-nearly compact. Then X is locally nearly compact.

<u>Proof.</u> By Theorem 2.3 in [4] Y is Hausdorff. Since Y is locally compact and Hausdorff it follows that, for each point $x \in X$ there is a closed compact neighbourhood V of f(x). Since f is almost continuous, the set $U = f^{-1}(\mathrm{Int}V)$ is open in X. By Theorem 1 in [8], the set $f^{-1}(V)$ is α-nearly compact in X. Since for each point $y \in Y$, $f^{-1}(y)$ is α-Hausdorff and the union of α-Hausdorff sets is α-Hausdorff, it follows that $f^{-1}(V)$ is α-Hausdorff. By Theorem 2.1 in [4] $f^{-1}(V)$ is closed. Now, we have

$$x \in U \subset \mathrm{Cl}U \subset f^{-1}(V).$$

Since every regularly closed subset of an α-nearly compact set is α-nearly compact, it follows that ClU is α-nearly compact. Now, U is an open neighbourhood of x such that ClU is α-nearly compact, hence X is locally nearly compact.

Corollary 2.2. ([8]) Let $f:X\to Y$ be a closed almost continuous surjection with N-closed point inverses. If X is Hausdorff and Y is locally compact, then X is locally nearly compact.

Theorem 2.6. Let f be an almost closed mapping of a space X onto a space Y. Let B be a closed subset of X such that for each $x\in X\setminus B$ the set $f^{-1}(f(x))$ is α-regular and α-paracompact with respect to B. Then, f(B) is closed.

Proof. Let

$$y\in Y\setminus f(B).$$

Then

$$f^{-1}(y)\subset X\setminus B.$$

By Theorem 2.6 in [6],

there is an open neighbourhood of $f^{-1}(y)$ such that

$$f^{-1}(y)\subset V\subset \mathrm{Cl}V\subset X\setminus B.$$

Since f is almost closed, then there is an open set W in Y such that $y\in W$ and $f^{-1}(y)\subset f^{-1}(W)\subset \mathrm{IntCl}V\subset X\setminus B$. Thus, we have $y\in W\subset Y\setminus f(B)$. Hence the statement.

Theorem 2.7. Let X be an R_0 space such that for each $x\in X$ $\mathrm{IntCl}(x)\neq\phi$. If $f:X\to Y$ is an almost closed mapping of the space X onto a space Y such that the family $\{f^{-1}(y):y\in Y\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then f is continuous.

Proof. Suppose that f is not continuous at some point $x\in X$. Let $\mathcal{U}(x)$ denote the family of all the open neighbourhoods of x. Let $y=f(x)$. Since f is not continuous at x, there is an open neighbourhood V of y such that

$$f(U) \cap (Y \setminus V) \neq \phi$$

for every $U \in \mathcal{U}(x)$. Thus,

$$\mathcal{A} = \{f(\mathrm{Cl}U) \cap (Y \setminus V) : U \in \mathcal{U}(x)\}$$

is a family of closed subsets of Y such that

$$\cap \{f(\mathrm{Cl}U) \cap (Y \setminus V) : U \in \mathcal{U}(x)\} \neq \phi$$

(X is R_o such that $\mathrm{IntCl}(x) \neq \phi$, for each $x \in X$. Thus, $U_o = \cap \{U : U \in \mathcal{U}(x)\}$ is an open set containing x and hence a member of $\mathcal{U}(x)$. So $(Y \setminus V) \cap f(U_o) \neq \phi$. i.e.

$$\cap \{A : A \in \mathcal{A}\} = f(\mathrm{Cl}U_o) \cap (Y \setminus V) \neq \phi).$$

Thus, there is a point $y_o \in \cap \{A : A \in \mathcal{A}\}$. Hence we have $y_o \in Y \setminus V$ and $x \notin f^{-1}(y_o)$. Since the family $\{f^{-1}(y) : y \in Y\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact, there are disjoint regularly open sets U_x and U_o such that

$$x \in U_x \text{ and } f^{-1}(y_o) \subset U_o.$$

From

$$\mathrm{Cl}U_x \cap f^{-1}(y_o) \subset \mathrm{Cl}U_x \cap U_o = \phi$$

we have

$$y_o \notin f(\mathrm{Cl}U_x).$$

On the other hand, since U_x belongs to $\mathcal{U}(x)$, we have

$$y_o \in f(\mathrm{Cl}U_x) \cap (Y \setminus V) \subset f(\mathrm{Cl}U_x).$$

This is a contradiction. Hence, f must be continuous at x.

Thus, f is continuous.

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REZIME

NEKE OSOBINE PODSKUPOVA I SKORO
ZATVORENIH PRESLIKAVANJA

U radu se ispituju neke osobine α-Hausdorfovih, α-regularnih i α-skoro regularnih podskupova topološkog prostora X. Daju se i uslovi kada je blizu parakompaktan prostor skoro regularan u prostoru koji ne mora da bude Hausdorfov. Daju se takodje i uslovi kada je skoro zatvoreno preslikavanje neprekidno nad prostorom koji ne mora da bude Hausdorfov.

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