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ON NUMERICAL SOLUTION OF A TURNING POINT PROBLEM

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Abstract

A numerical method for a singular perturbation problem with a turning point is considered. The method uses non-equidistant discretization meshes. The first order accuracy, uniform in the perturbation parameter is proved in the discrete L^1 norm.

1. Introduction

In this paper we consider a numerical method for the following singularly perturbed boundary value problem with a turning point:

(1a) $$\varepsilon^2 u'' + xb(u)u' = f_\varepsilon(x), \quad x \in I = [a,1],$$

(1b) $$u(a) = A, \quad u(1) = B$$

with $a=0$ or $a=-1$. By ε we denote the perturbation parameter: $0 < \varepsilon \le 1$ (usually $\varepsilon \ll 1$). The functions b, f_ε and numbers A, B are given. Our basic assumptions are:

(2a) $$b,\ f_\varepsilon \in C^1(I),$$

(2b) $$\beta := \min_{x\in I} b(x) > 0,$$

(2c) $$|f_\varepsilon(x)| \le M(|x| + F_\varepsilon(x)), \quad x \in I,$$

(2d) $$|f'_\varepsilon(x)| \le M(1 + (\varepsilon^{-2}|x| + \varepsilon^{-1})\, F_\varepsilon(x)), \quad x \in I,$$

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where

$$F_\varepsilon(x) = exp(-\int_0^x c(t)dt/\varepsilon^2), \ c(x) := xb(x).$$

Here and throughout the paper M denotes any positive constant, independent of ε. For simplicity, in the rest of the paper the dependence of f_ε and F_ε on ε will not be denoted. Furthermore, we require:

(3) $$b'(x) > -3\beta/2, \quad x \in I.$$

We shall begin our investigation with the case $a=0$, which will be considered in Section 2. In Section 3 we shall give necessary modifications concerning the case $a=-1$.

Section 2 contains several parts. In the first part we investigate problem (1) and we give estimates for the derivatives of its solution, u. We use the assumption (2) to prove the appropriate estimates. For instance, (2b,c) guarantee that u is bounded uniformly in ε. Note that (2c,d) allow the case $f\equiv0$, for which the asymptotic behaviour of the solution has usually been investigated, [8]. Since u has an $O(\varepsilon)$ boundary layer at $x=0$, a special numerical method should be applied. We shall use finite-differenc schemes on a special non-equidistant mesh which is dense in the layer. The mesh, I_h, (with the mesh points $0=x_0<x_1<\ldots<x_n=1$, $n \in N$), is generated by a suitable function λ (i.e. $x_i = \lambda(ih)$, $h=1/n$). λ depends on ε in such a way that the smaller ε becomes, the more the mesh is condensed in the layer. Essentially, λ is a certain modification of the inverse of the boundary layer function. The finite-difference discretization and its stability are discussed in the second part of Section 2. The stability is proved in the discrete L^1 norm. The assumption (3) is important for our stability result. In the third part of Section 2 we deal with the consistency error using the derivative estimates from the first part and properties of function λ. In this part constants M are independent of h as well. Our basic result is

(4) $$\|w_h - u_h\|_h \le h.$$

Here and throughout the paper we use the following notation:

$w_h = [w_1, w_2, \ldots, w_{n-1}]^T \in R^{n-1}$ is the numerical solution,

$u_h = [u(x_1), u(x_2), \ldots, u(x_{n-1})]^T \in R^{n-1}$ is the discretization of the continuous solution,

$$\|z_h\|_h = \sum_{i=1}^{n-1} \bar{h}_i |z_i|, \quad z_h = [z_1, z_2, \ldots z_{n-1}]^T \in R^{n-1} ,$$

$\bar{h}_i = (h_i + h_{i+1})/2$, $h_i = x_i - x_{i-1}$. We can think of (4) as a first order convergences uniform in ε (which is the final aim of all numerical methods for singular perturbation problems) even though $\|\cdot\|_h$ depends on ε since h_i's depend on it. Still, $\|\cdot\|_h$ is the usual discrete L^1 norm on the non--equidistant mesh, cf. [1]. We end Section 2 by presenting some numerical results.

Up till now, numerical methods for linear turning point problems have usually been considered in the case when the left hand side of (1a) contains the additional term $d(x)u$, where $d(x)>0$, $x \in I$, or at least $d(0)>0$, see [3], [4], [5], [7]. In the first three of these papers equidistant discretizations only are considered and upwind or exponentially fitted schemes are used. Paper [7] uses mesh generation - the approach which we shall apply here. This approach dates from 1969, [2], where a self-adjoint singularly perturbed boundary value problem was considered, and it has been modified and applied to other types of problems with a small parameter, see [7], [9], [10], [11], for instance. In [12], the authors deal with problems of type (1), among others, investigating the ill conditioning of the corresponding exponentially fitted discretization on the equidistant mesh.

2. Case a=0.

Throughout this Section we shall consider problem (1) with a=0, assuming that (2) holds.

2.1 Analysis of the Continuous Problem

For the proof of the following lemma we do not need the assumption (2d):

Lemma 1. *Problem (1) has a unique solution $u \in C^3(I)$ which is bounded uniformly in ε:*

$$|u(x)| \le M, \ x \in I.$$

Proof. Consider the operators

$$(5) \qquad Lu := -\varepsilon^2 u'' - xb(x)u' \ , \ Ru := (u(0), \ u(1)).$$

Since (L,R) is inverse monotone,uniqueness of the solution is guaranteed. Existence and boundedness uniform in ε follow because there exist unifomly bounded upper and lower solutions to the problem (1). To show that use

$$v(x) = M_1(2-x) + M_2F(x)$$

with appropriate constants M_1, M_2 (independent of ε), such that $v(0) \ge |A|$, $v(1) \ge |B|$ and

$$Lv(x) = M_1 xb(x) + M_2c'(x)\ F(x) \ge |f(x)|. \tag{6}$$

The inequality in (6) can be achieved because of (2b,c). To see this, note that there exists a number $\delta \in (0,1]$ (independent of ε), such that $c'(x) \ge \gamma > 0$ for $x \in [0,\delta]$. Then, if $0 \le x \le \delta$ use

$$Lv(x) \ge M_1\beta x + M_2\gamma F(x) \ge |f(x)|.$$

On the other hand, if $\delta \le x \le 1$:

$$Lv(x) \ge M_1\beta x + M_2c'(x)\ F(x) \ge |f(x)|.$$

Hence, the upper solution is $v(x)$ and the lowe one is $-v(x)$. □

Lemma 2. $|u^{(i)}(0)| \le M\varepsilon^{-i}$, i=1,2 .

Proof. In the case i=2, the proof follows from (1a). For i=1 rewrite (1a) in the form

$$\varepsilon^2u'' + (cu)' = f + c'u$$

and integrate this equation from 0 to x_*, where $x_* \in (0,\varepsilon)$ is such a point that $u'(x_*) = (u(\varepsilon) - u(0))/\varepsilon$ (hence $|u'(x_*)| \le M/\varepsilon$). □

Lemma 3. $|u'(x)| \le M(1 + (x\varepsilon^{-2} + \varepsilon^{-1})\ F(x))$, $x \in I$.

Proof. For the technique cf. [6]. Rewrite (1a) as follows:

$$\varepsilon^2(u'/F)' = f/F$$

and express u' by integration. Then because of (2c) and Lemma 2 we have:

$$|u'(x)| \le M(S + \varepsilon^{-1}F(x)),$$

where

$$S = \varepsilon^{-2}\int_0^x (t + F(t))\ (F(x)/F(t))dt.$$

Since

$$\int_x^t sb(s)ds \le \beta(t^2-x^2)/2 \ , \ 0 \le t \le x,$$

it follows

$$S \le x\varepsilon^{-2}F(x) + \varepsilon^{-2}\int_0^x t \ exp(0.5\beta(t^2-x^2)/\varepsilon^2)dt \le x\varepsilon^{-2}F(x) + M \ . \ \square$$

Lemma 4. $|xu''(x)| \le M(1 + \varepsilon^{-1}exp(-\beta x^2/(4\varepsilon^2)))$, $x \in I$.

Proof. Differentiate (1a) and obtain

$$\varepsilon^2(u''/F)' = (f'-c'u')/F \ .$$

Then because of (2d) and Lemmas 2 and 3:

$$|u''(x)| \le M(\varepsilon^{-2} \int_0^x (F(x)/F(t))dt + (x^2\varepsilon^{-4} + x\varepsilon^{-3})F(x) + \varepsilon^{-2}F(x)) \ .$$

Since

$$x\varepsilon^{-2} \int_0^x (F(x)/F(t))dt \le x\varepsilon^{-2} \int_0^x exp(0.5\beta(t-x)x/\varepsilon^2)dt \le M \ ,$$

and

$$x^k\varepsilon^{-(k+1)}F(x) \le M\varepsilon^{-1}exp(-\beta x^2/(4\varepsilon^2)) \ , \quad k=1,2,3 \ ,$$

the lemma is proved. $\square$

Using previous lemmas, we can obtain the following estimates which will be used in 2.3 in the analysis of the consistency error.

Theorem 1. *For $x \in I$ we have*

(7a) $|(cu)''(x)| \le M(1+\varepsilon^{-1}y(x))$,

(7b) $\varepsilon^2|u'''(x)| \le M(1+\varepsilon^{-1}y(x))$,

(7c) $\varepsilon^2|u''(x)| \le M(|x| + y(x))$,

(7d) $|(cu)'(x)| \le M$,

where $y(x) = exp(-|x|/\varepsilon)$.

Proof. Note that for any $\sigma>0$ independent of ε it holds that

$$exp(-\sigma x^2/\varepsilon^2) \le M\, y(x) .$$

Then the inequalities (7a) and (7d) follow easily from Lemmas 3 and 4, while (7c) follows directly from (1a). Finally, differentiate (1a) and use Lemmas 3 and 4 to obtain (7b). □

2.2 The Discretization and Its Stability

Note that

$$\|z_h\|_h = \|Hz_h\|_1$$

where $H= diag(\bar{h}_1, \bar{h}_2, \dots \bar{h}_{n-1}) \in R^{n-1,n-1}$ and $\|\cdot\|_1$ is the usual vector norm in R^{n-1}. The corresponding matrix norm is:

$$\|A_h\|_h = \|HA_hH^{-1}\|_1 ,$$

where A_h is a matrix in $R^{n-1,n-1}$ and $\|\cdot\|_1$ is the usual matrix norm in that space.

The discrete operator on the mesh I_h is:

$$L_hz_i := -\varepsilon^2 D''_h z_i - D'_h c(x_i)z_i + c'(x_i)z_i$$

and it corresponds to L from (5) which is rewritten in the conservative form

$$Lu = -\varepsilon^2 u'' - (cu)' + c'u .$$

Here

$$D''_h z_i = (h_{i+1}z_{i-1} - 2\bar{h}_i z_i + h_i z_{i+1})/(h_i h_{i+1} \bar{h}_i) ,$$

and

$$D'_h z_i = (z_{i+1} - z_i)/\bar{h}_i ,$$

cf. [1]. Thus, the discrete problem reads:

(8)
$$L_h w_i = -f(x_i) , \quad i=1,2,\dots n-1 ,$$
$$w_0 = A , \quad w_n = B .$$

Rewrite (8) in the matrix form:

(9)
$$A_h w_h = d_h ,$$

where $d_h = [d_1, d_2, \dots d_{n-1}]^T \in R^{n-1}$, $d_i = -f(x_i)$, $i=2,3,\dots n-2$, $d_1 = -f(x_1) + \varepsilon^2 A/(h_1\bar{h}_1)$, $d_{n-1} = -f(x_{n-1}) + \varepsilon^2 B/(h_n \bar{h}_{n-1}) + c(x_n)B/\bar{h}_{n-1}$,

and A_h is the corresponding tridiagonal matrix.

Now we shall prove stability of the discrete problem (9) in the norm $\|\cdot\|_h$.

Theorem 2. *Let* $b \in C^1(I)$, $c'(x) \ge \gamma > 0$, $x \in I$, *and let* (2b) *hold. Then:*

(10)
$$\|A_h^{-1}\|_h \le M .$$

Proof. Because of the upwind scheme it is easy to see that A_h is an L-matrix (the diagonal elements are positive and the off diagonal elements are non-negative.) Furthermore:

$$(HA_hH^{-1})^T e_h \ge \gamma e_h ,$$

where $e_h = [1,1,\ldots,1]^T \in R^{n-1}$. Thus A_h is an M-matrix ($A_h^{-1} \ge 0$) and the result follows. □

However, we can show that the first condition of Theorem 2 can be weakened and replaced by (3), provided that $h=1/n$ be sufficiently small (but independent of ε). For that purpose we have to specify our discretization mesh. Let us define the mesh generating function λ:

(11)
$$\lambda(t) = \begin{cases} \omega(t) := Q\varepsilon t/(q-t) , & t \in [0,\alpha] \\ \pi(t) , & t \in [\alpha ,1] \end{cases} .$$

Here $\alpha \in (0,1)$ is an arbitrary parameter (independent of ε),

(12)
$$q = \alpha + \varepsilon^{1/3} ,$$

and $\pi(t)$ is a third order polynomial, such that $\lambda \in C^2(I)$ and $\pi(1)=1$. The parameter Q should be positive and chosen in such a way that $\pi''' \ge 0$ (a simple analysis shows that $Q\in(0,1/2]$ would suffice for all $\varepsilon\in(0,1]$ and $\alpha \in (0,1)$). This implies

$$\pi^{(k)}(t) \ge \pi^{(k)}(\alpha) = \omega^{(k)}(\alpha) > 0 , \quad t \in [\alpha,1],$$

firstly for $k=2$ and then for $k=1$. Obviously:

$$\omega^{(k)}(t) > 0 , \quad k=0,1,\ldots, \quad t \in [0,\alpha] ,$$

and taking (12) into account:

(13a)
$$0 < \lambda^{(k)}(t) \le M , \quad k=1,2 , \quad t \in I.$$

Furthermore, note the inequality:

(13b)
$$exp(-\omega(t)/\varepsilon) \le M\, exp(-M/(q-t)) , \quad t \in (0,q) ,$$

which will be used in Section 2.3.

The mesh points are given by

(14) $$x_i = \lambda(t_i)\ ,\ t_i = ih\ ,\ i=0,1,\dots,n\ .$$

There are other possible choices of function λ, cf. [9], but because of simplicity, (1) is the only form which we shall consider here.

Theorem 3. *Let $b \in C^1(I)$ and let (2b) and (3) hold. Then the discretization matrix A_h on the mesh (14), (11), with sufficiently small h and Q, independent of ε, satisfies (10).*

Proof. Let $e_h = [1+x_1,\ 1+x_2, \dots,\ 1+x_{n-1}]^T \in R^{n-1}$. Then for $i=1,2,\dots,n-1$ we have:

$$((HA_hH^{-1})^T e_h)_i \geq (1+x_i)c'(x_i) + (h_i/\bar{h}_i)c(x_i) =: s_i\ .$$

Let δ and γ have the same meaning as in the proof of Lemma 1.

Then if $x_i \leq \delta$:

$$s_i \geq \gamma > 0\ .$$

Now let $x_i > \delta$. We shall prove

(15) $$s_i \geq \gamma_1 > 0\ ,$$

for some γ_1 independent of ε and h, and the proof will be completed. Rewrite s_i in the form:

$$s_i = s_i' - p_i,$$

where

$$s_i' = (1+x_i)c'(x_i) + c(x_i)\ ,$$

$$p_i = ((h_{i+1}-h_i)/(h_{i+1}+h_i))\ c(x_i) > 0\ .$$

Note that by (3) there exists a constant $\mu > 0$ (independent of ε), such that $b'(x) \geq \mu - 3\beta/2$. Then:

$$s_i' \geq \mu(1+x_i)x_i + (\beta/2)(-3x_i^2+x_i+2) > \mu(1+\delta)\delta\ .$$

Next we shall show that

(16) $$p_i \leq Mh$$

and (15) will follow provided that h be sufficiently small. Let us first note that

$$\pi(t) = a_3(t-\alpha)^3 + a_2(t-\alpha)^2 + a_1(t-\alpha) + a_0\ ,$$

where

$$a_0=\omega(\alpha)=Q\alpha\varepsilon^{2/3}\ ,\quad a_1=\omega'(\alpha)=Qq\varepsilon^{1/3}\ ,\quad a_2=\omega''(\alpha)/2=Qq\ ,$$
$$a_3=(1-Q(q(1-\alpha)^2-q(1-\alpha)\varepsilon^{1/3}-\alpha\varepsilon^{2/3}))/(1-\alpha)^3\ .$$

Then it is easy to see that there exist sufficiently small parameter Q and sufficiently small constant $m > 0$ (both independent of ε), such that

$$\pi(\alpha+m) = \lambda(\alpha+m) \le \delta < m_1 = \lambda(t_1)\ .$$

This implies $t_1>\alpha+m$ and $t_{1-1}\ge\alpha+m$, if h is small enough. Then because of (13a) we have

$$p_1\le M\ h^2\lambda''(t_{1+1})/(2h\lambda'(t_{1-1}))\ ,$$

and (16) follows from (13a) and $\lambda'(t_{1-1})\ge\lambda'(\alpha+m)>2Qqm$. □

Remark. The requirement in Theorem 3 that the parameter Q should be sufficiently small is neeeded to make the coefficient $a_0=Q\alpha\ \varepsilon^{2/3}$ sufficiently small. However, if ε is small enough, there is no need for such a constraint on the parameter Q.

2.3 The Convergence Result

Let us consider the consistency error

$$r_1 := L_h u(x_1)-(Lu)(x_1)\ ,\quad i=1,2,\dots,n-1\ .$$

We have

$$r_1 = r_1'' + r_1'\ ,$$
$$r_1'' = \varepsilon^2(u''(x_1)-D_h''u(x_1))\ ,\quad r_1' = g'(x_1)-D_h'g(x_1)\ ,$$

where $g(x):= c(x)u(x)$. Let $r_h= [r_1,r_2,\dots,r_{n-1}]^T\in R^{n-1}$. Then we have:

Theorem 4. *Let* (2) *hold. Let discrete operator* L_h *be given on the mesh* (14), (11). *Then* $\|r_h\|_h\le Mh$.

Proof. We shall prove

(17a) $$|r_1''| \le Mh\ ,$$

(17b) $$h_{1+1}|r_1'| \le Mh^2\ ,$$

and the result will follow. The following estimates hold:

(18a) $$|r_1''| \le M\varepsilon^2 h_{1+1}|u'''(\sigma_1'')|\ ,\quad \sigma_1'' \in (x_{1-1},x_{1+1})\ ,$$
$$|r_1''| \le 2\varepsilon^2 \max_{x_{1-1}\le x\le x_{1+1}} |u''(x)|\ ,$$

$$|r_i'| \le ((h_{i+1}-h_i)/(h_i+h_{i+1}))|g'(\sigma_i')| + G_i/h_{i+1} ,$$
$$\sigma_i' \in (x_i, x_{i+1}) ,$$

where

$$G_i = \int_{x_i}^{x_{i+1}} (x_{i+1}-x)|g''(x)|dx . \tag{19}$$

To prove (17b) it is sufficient to show

$$G_i \le Mh^2 , \tag{20}$$

since from (7d) and (13a) we have

$$h_{i+1}((h_{i+1}-h_i)/(h_i+h_{i+1}))|g'(\sigma_i')| \le M(h_{i+1}-h_i) \le Mh^2 .$$

Let us prove (17a) and (20). We shall use the technique which is essentially the one from [2], cf. [7], [9], [10], [11]. The proof is devided into three steps:

1° $t_{i-1} \ge \alpha - \varepsilon^{1/3}$,

2° $t_{i-1} \le \alpha - 3h$,

3° $\alpha - 3h < t_{i-1} < \alpha - \varepsilon^{1/3}$

1° From (18a), (13) and (7b) we have:

$$|r_i''| \le Mh(1+\varepsilon^{-1} y(x_{i-1})) \le Mh(1+\varepsilon^{-1}y(\lambda(\alpha-\varepsilon^{1/3}))) \le Mh .$$

Similarly, from (19), (7a) and (13) it follows

$$G_i \le Mh^2(1+\varepsilon^{-1} y(x_i)) \le Mh^2 .$$

2° In this case we have $\alpha - t_{i+1} \ge (\alpha - t_{i-1})/3$ and thus $q-t_{i+1} > (q-t_{i-1})/3$. Now (18a), (13) and (7b) give

$$|r_i''| \le Mh(1+(q-t_{i+1})^{-2} y(\lambda(t_{i-1}))) \le Mh .$$

In the same way:

$$G_i \le Mh^2(1+(q-t_{i+1})^{-4} y(\lambda(t_i))) \le Mh^2 .$$

3° Now $\varepsilon^{1/3} < 3h$. We use (18b) and (7c) to obtain

$$|r_i''| \le M(\lambda(t_{i+1}) + y(\lambda(t_{i-1}))) \le M(\lambda(\alpha) + h\lambda'(t_{i+1}) + y(\lambda(\alpha-3h))) \le Mh.$$

On the other hand

$$G_i \le M \int_{x_i}^{x_{i+1}} (x_{i+1}-x)(1+\varepsilon^{-1} y(x))dx ,$$

and after integration:

$$G_i \le M(h^2+ hy(x_i)) \le M(h^2+ hy(\lambda(\alpha-3h))) \le Mh^2 . \square$$

Thus, combining Theorems 2 and 3 with Theorem 4, we can get the corresponding convergence results:

Theorem 5. *Let u be the solution to the continuous problem* (1) *and let* (2) *and* $c'(x)\ge\gamma>0$, $x \in I$, *hold. Let* w_h *be the solution to the discrete problem* (8) *on the mesh* (14), (11). *Then we have:*

(21) $$\|w_h-u_h\|_h\le Mh .$$

Theorem 6. *Let u be the solution to the continuous problem* (1) *and let* (2) *and* (3) *hold. Let* w_h *be the solution to the discrete problem* (8) *on the mesh* (14), (11) *with sufficiently small h and Q, independent of* ε. *Then* (21) *holds.*

2.4 Numerical Results

We shall present results concerning the following test problem:

$$-\varepsilon^2u'' - 2xu'= 2exp(-(x/\varepsilon))^2 , \ u(0)=1 , \ u(1)= exp(-1/\varepsilon^2) ,$$

whose solution is known: $u(x) = exp(-(x/\varepsilon)^2)$. Table 1 contains the results obtained by using the mesh generating function (11) with paremeters $Q=1$, $\alpha=0.5$. By changing these parameters we can change the density of the mesh in the layer. In this case the percentage of the mesh points which lie within the layer varies from about 25% to about 35% (it changes slightly when ε and n change).

TABLE 1.

ε	n	50	100	200
1.-2	E	1.47-2	7.54-3	3.82-3
	E_h	2.08-4	1.06-4	5.40-5
1.-3	E	1.72-2	8.94-3	4.54-3
	E_h	2.46-5	1.26-5	6.42-6
1.-4	E	1.88-2	9.79-3	4.98-3
	E_h	2.67-6	1.38-6	7.04-7

We use the following notation:

$$E= \|w_h-u_h\|_\infty , \quad E_h= \|w_h-u_h\|_h ,$$

and, as usual, 1.-2 means 10^{-2}etc.

Table 1 shows more than our theory gives: the first order pointwise convergence uniform in ε can be observed, and the error E_h decreases when ε does. However, we can't expect this in general. Some other problems, which we have tested, show the first order L^1 convergence uniform in ε only, which coincide with our theory.

3. The Case a=-1

Now $I=[-1,1]$. First we shall show that Theorem 1 holds in this case as well. The only difference is in the proof of Lemma 1, where the function $v(x)$ should take the form:

$$v(x)=M_1(2-p(x)) + M_2F(x) ,$$

where $p(x)$ is a $C^2(I)$-function, having the following properties: $p(x)=|x|$, $x \in I\setminus[-\delta,\delta]$; $p''(x) > 0$, $xp'(x) \geq 0$, $x \in (-\delta,\delta)$.

Next, the discretization mesh (14) should be used with

$$t_i=-1+ih,\ i=0,1,\dots,n\ ,\ h=2/n\ ,\ n=2k\ ,\ k \in \mathbb{N}\ .$$

The mesh generating function coincides on [0,1] with λ from (11), and for $t \in [-1,0]$ we take $\lambda(t)= -\lambda(-t)$. Thus, λ'' is discontinuous at $t=0$, but this will not affect the proof of Theorem 4 since the two mesh steps in vicinity of $x=0$ are equidistant. At the mesh points x_i , $i=1,2,\dots,k-1$, the scheme L_h should be used with

$$D'_h z_i = (z_i - z_{i-1})/\bar{h}_i \ .$$

Finally, Theorem 2 can be proved analogously, thus Theorem 5 follows too.

Remark. This paper contains the results of the first stage of investigations of the problems of type (1). We expect to improve these results in forthcoming papers.

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Rezime

O NUMERIČKOM REŠAVANJU PROBLEMA SA POVRATNOM TAČKOM

Posmatra se numerički metod za singularni perturbacioni problem sa povratnom tačkom. Metod koristi neekvidistantne mreže disktetizacije. U diskretnoj normi L^1 je dokazan prvi red tačnosti, uniforman po perturbacionom parametru.

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