Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 19,1,109–117 (1989)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

A NOTE ON α-REGULARITY AND COMPACTNESS

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ABSTRACT:

The aim of the paper is to study some properties of α-nearly compact (nearly compact), α-almost compact (almost compact) and compact sets (spaces) in topological spaces which are not Hausdorff or regular (almost regular). Some results concerning compactnessu are generalized.

1. PRELIMINARIES

Throughout the present paper, spaces will always mean topological spaces on which no sparation axioms are unless explicitly stated.

A subset of a spece X is *regularly open (regularly closed)* iff it is an interior (closure) of some closed (open) set or equivalently iff it is an interior (clo-

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AMS Mathematics Subject Classification (1980): 54D10, 54D30

Key words and phrases: α-nearly compact, α-almost compact, α-regular, α-almost regular, α-Hausdorff.

sure) of its own closure (interior), [1] . A space X is *nearly compact* iff every regularly open cover has a finite subcovering, [5].

A space X is *almost compact* iff for every open cover $\{U_i : i\in I\}$ there exists a finite subfamily I_o of I such that $X=\cup\{\overline{U}_i : i\in I_o\}$, [4].

A subset A of a space X is said to be *α-nearly compact* (N-closed) iff for every regularly open cover $\mathcal{U}=\{U_i : i\in I\}$ of A there exists a finite subset I_o of I such that $A\subset\cup\{U_i : i\in I_o\}$, [6].

A subset A of a space X is said to be *α-almost compact* (H-closed) iff for every open cover $\mathcal{U}=\{U_i : i\in I\}$ of A there exists a finite subset I_o of I such that $A\subset\cup\{\overline{U}_i : i\in I_o\}$, [4].

A subset A of a space X is *α-Hausdorff* iff any two points a,b of a space X, where $a\in A$ and $b\in X\setminus A$, there are disjoint open sets U and V containing a and b respectively, [3].

A subset A of a space X is *α-regular (α-almost regular)* iff for any ponit $a\in A$ and any open (regularly open) set U containing a there exists an open set V such that $a\in V\subset\overline{V}\subset U$ or equivalently, for any closed (regularly closed) set F and any point $x\in A$ such that $x\in X\setminus F$, there exist disjoint open neighbourhoods of x and F respectively, [3], ([2]).

A space X is *almost regular* iff for any regularly closed set F and any point $x\notin F$, there exist disjoint open sets containing F and x respectively, [7].

2. SOME RESULTS

Theorem 2.1. If A is an α-regular and compact subset of a space X, then $\overline{A}$ is compact.

Proof. Let

$$\mathcal{U}=\{U_i : i\in I\}$$

be any open covering of A. For each $x\in A$, there exists an open set U_x containing x. There exists an open set V_x such that

$$x\subset V_x\subset \overline{V_x}\subset U_x.$$

Let

$$\mathcal{V}=\{V_x : x\in A\}.$$

There exists a finite points $x_1, x_2, \ldots, x_n$ in A such that

$$A\in \bigcup_{i=1}^{n} V_{x_i}$$

Thus we have

$$\overline{A}\subset \overline{\bigcup_{i=1}^{n} V_{x_i}} = \bigcup_{i=1}^{n} \overline{V}_{x_i} \subset \bigcup_{i=1}^{n} U_{x_i} .$$

Hence $\overline{A}$ is compact.

Theorem 2.2. If A is an α-almost regular and α-nearly compact subset of a space X, then $\overline{A}$ is α-nearly compact.

Proof. It is similar to the proof of Theorem 2.1. The closure of an α-regular and compact (α-almost regular and α-nearly compact) subset is not always α-regular (α-almost regular). The following example serves the purpose.

Example 2.1. Let

$$X=\{a,b,c,a_i : i=1,2,\ldots\}$$

Let

$$A=\{b,a_i : i=1,2,\ldots\}$$

Let each point a_i be isolated. Let a point c be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a):n=1,2,\ldots\},$$

where

$$V^n(a)=\{a,a_i:i\geq n\}.$$

Let the fundamental system of neighbourhoods of b be the set

$$\{U^n(b):n=1,2,\ldots\},$$

where

$$U^n(b)=V^n(a)\cup\{b,c\}.$$

The set A is α-regular and compact.

$$\overline{A}=\{a,b,a_i:i=1,2,\ldots\}$$

is compact, but is not α-almost regular (hence $\overline{A}$ is not α- regular) (for any regularly open neighbourhood $V^n(a)$ of a $\overline{V^n(a)}=V^n(a)\cup\{b\}$).

The closure of a regular and compact subset is not always compact, as we can see from the following example.

Example 2.2. Let

$$X=\{a,a_i:i=1,2,\ldots\}$$

Let a point a be isolated. For each a_i let the fundamental system of neighbourhoods be the set

$$\{a,a_i\}.$$

The set

$$A=\{a\}$$

is regular and compact, but $\overline{A}$=x is not compact (A is not α-regular).

The following example shows that there is a dense α-regular (α-almost regular) and compact (α-nearly compact) subset of a space X, which is not regular (almost regular)

Example 2.3. Let X be a space in example 2.1. Let

$$A=\{b,c,a_i:i=1,2,\ldots\}$$

A is a dense α-regular and compact subset of a spece X. X is not almost regular (regular) at a, hence X is not almost regular (regular).

Theorem 2.3. Every α-regular α-almost compact is compact.

<u>Proof</u>. Let

$$\mathcal{U}=\{U_i:i\in I\}$$

be any open cover of an α-regular α-almost compact subset A. For each $x\in A$ there exists an $i(x)\in I$ such that $x\in U_{i(x)}$. There exists an open set V_x such that

$$x\in V_x\subset \bar{V}_x\subset U_{i(x)}.$$

Let

$$\mathcal{V}=\{V_x:x\in A\}.$$

Since $\mathcal{V}$ is an open cover of A, there exists a finite subfamily

$$\{V_{x_j}:j=1,2,\ldots,n\}$$

such that

$$A\subset \bigcup_{j=1}^{n} \bar{V}_{x_j} .$$

Then,

$$\{U_{i(x_j)}:j=1,2,\ldots,n\}$$

is a finite subcover of $\mathcal{U}$, hence A is compact.

Theorem 2.4. Every α-almost regular α-almost compact is α-nearly compact.

Proof. It is similar to the proof of Theorem 2.3.

Corollary 2.1. Every α-ragular α-nearly compact is compact.

Proof. Every α-nearly compact is α-almost compact.

Theorem 2.5. If in a space X there exists a dense α-regular (α-almost regular) compact (α-nearly compact), then X is compact (nearly compact).

Proof. The closure of α-regular compact (α-almost regular α-nearly compact) is compact (α-nearly compact).

Theorem 2.6. Let A be any dense α-regular subset of a space X such that every open cover of A is an open cover of X. X is almost compact iff it is compact.

Proof. If X is compact, then X is almost compact (every compact is almost compact). Let X be almost compact. Let

$$\mathcal{U}=\{U_i : i\in I\}$$

be any open covering of X. For each point $x\in A$ there exists $i(x)\in I$ such that $x\in U_{i(x)}$. Since A is α-regular, there exists an open set V_x such that

$$x\in V_x \subset \overline{V}_x \subset U_{i(x)}.$$

Let

$$\mathcal{V}=\{V_x : x\in A\}.$$

$\mathcal{V}$ is an open cover of A i.e. of X. Since X is almost compact there exists a finite subfamily

$$\{V_{x_j} : j=1,2,\ldots,n\},$$

such that

$$X = \bigcup_{j=1}^{n} \bar{V}_{x_j}$$

Then,

$$\{U_{i(x_j)} : j=1,2,\ldots,n\}$$

is a finite subcovering of $\mathcal{U}$, hence X is compact. (A is compact).

Theorem 2.7. Let A be any dense α-almost regular subset of a space X such that every regularly open cover of A is a regularly open cover of X. X is almost compact iff it is nearly compact.

Proof. It is similar to the proof of Theorem 2.6. In Theorem 2.6. we assumed that every open cover of a dense α- regular subset is an open cover of X. This assumption cannot be dropped, as can be seen from the following example.

Example 2.4. Let

$$X=\{a_{ij}, a_i, a : i,j=1,2,\ldots.\}$$

Let each point a_{ij} be isolated. Let the fundamental system of neighbourhoods of a_i be the set

$$\{U^n(a_i) : n=1,2,\ldots\},$$

where

$$U^n(a_i)=\{a_i, a_{ij} : j>n\}.$$

Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a): n=1,2,\ldots\},$$

where

$$V^n(a)=\{a, a_{ij}: i>n, j>n\}.$$

X is a Hausdorff space which is not regular at a point a, hence X is not compact (every Hausdorff compact is regular). X is almost compact.
Let

$$A=\{a_{ij}, a_i\}\ .$$

A is an open α-regular subset of X. A is a dense subset.
A is not compact ($\bar{A}$=X is not compact).
We know that every regular almost compact is compact.

Example 2.3. shows that there exists a space with properties as in Theorem 2.6., which is not regular.
This example shows that thare exists a pace with properties as in Theorem 2.7., which is not almost regular.

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REZIME

α - REGULARNOST I KOMPAKTABILNOST

U radu se ispituju neke osobine α-regularnih odnosno α-skoro regularnih skupova povezanih sa kompaktnošću. Dokazuje se ekvivalentnost skore kompaktnosti sa kompaktnošću (blizu kompaktnošću) u prostorima koji nisu regularni (skoro regularni).

Received by the editors November 24, 1987.