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SOME ACHIEVEMENT AND AVOIDANCE GAMES ON PARTITIONAL MATROIDS

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Abstract

The paper solves (1-color-) achievement and avoidance (2-person-) games for bases, circuits and hyperplanes of partitional matroids. The solution of achievement game for bases is partial. Some solutions do hold for general matroids as well.

1. Preliminaries

An n-set is a set of cardinality n.

The reader is refered to, e.g., [9] for non-defined notions from matroid theory.

We shall consider the following achievement and avoidance games on hypergraphs:

Given a finite set S and a family F of its subsets, two players alternately choose elements from S and all these chosen elements accumulate. The game is over after the first move, which makes the chosen set contain a subset from F. The player who makes this last move is the winner in the achievement game, while he is the loser in the avoidance game.

Remarks:

If the family F contains some comparable sets, then the games should be played on the minimal antichain included in F.

If the family F contains a 1-set, then the first player trivially wins in Achieve F. Therefore we shall assume that singletons do not exist with families F on which the achievement games are played.

Similar games were considered, for example, in papers [1] - [7].

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2. Exact achievement games

Let (S,F) denote a general hypergraph. Assuming that its ground-set S is fixed, we introduce the following two families as the functions of F:

Alfa(F) = the family of maximal subsets of S which do not contain a set from F.

Beta(F) = the family of subsets X of S which satisfy: X does not contain a set from F, but there exists an element z in S and a set Y in F, such that $X\cup\{z\}$ contains Y.

We say that a set from F is *exactly achieved* if the last move of the game makes a set from F, but not a superset of a set from F. We can define an exact achievement game in a natural way.

Theorem. *The following pairs of games are equivalent:*

(a) Avoid F *and* Achieve exactly Alfa(F)

(b) Achieve F *and* Achieve exactly Alfa(Beta(F))

Proof. (a) It is obvious that a player is forced to achieve a set from F if and only if his opponent has produced a set from Alfa(F) in the previous move.

(b) If a player makes a set from Beta(F), then his opponent is able to achieve a set from F in the next move. The converse statement is also true. We conclude that the games Achieve F and Avoid (Beta(F)) are equivalent. There remains to apply (a). #

The analysis of achievement and avoidance games can be made easier by their reduction to exact achievement games. For example, if all the sets in Alfa(F) have the same cardinality, then the outcome of Avoid F can be immediately determined.

Problem. Give a good algorithm for solving Achieve exactly F.

3. Games on general matroids

We shall mostly restrict our attention to the case when the family F stands for either circuits or bases or hyperplanes of a matroid on S. Our short denotations for the corresponding achievement and avoidance games will be:

ACH CIRC, ACH BASE, ACH HYP, AV CIRC, AV BASE, AV HYP.

Any of these games is drawn only if the family F is empty of $F = \{\emptyset\}$. There are four such situations: F = circuits and matroid is free; F = bases and rank = 0; F = hyperplanes and rank= 0; F = hyperplanes = $\{\emptyset\}$ and rank= 1.

Given a matroid and a game, the outcome depends solely on the parity of the integer L (length of the game), where

L = the minimal number of moves in which one of the players (who wants to) can force the game to an end.

Theorem. *The game* AV CIRC *has a trivial general solution:*

$$L = (\text{rank of matroid}) + 1$$

Proof. Each non-final move must make a rank increase (by 1) of the chosen set. Base achievement wins. #

Remark. The length of ACH CIRC is obviously bounded from above by the same L.

It is obvious that Alfa(circuits) = bases and Alfa(bases) = hyperplanes. It follows that the games AV CIRC and ACH EXACT BASE, respectively AV BASE and ACH EXACT HYP, are equivalent.

If F is not a covering family for S, then each element from $S-F$ can be used for "prolonging" the game. For example, if we add x loops in the cases of base or hyperplane games, or x coloops in the case of ACH CIRC, then the length of the same game on thus generated matroids will become equal to $L + x$.

We shall give another example of the "prolonging strategy":

Let (n,r,c) denote the rank r matroid on n elements obtained from a uniform matroid by addition of c coloops (we assume that $n > r > c$). If ACH BASE is played on (n,r,c), then $L = r$ or $L = n$, depending on which one of the following two subgoals is achieved first: choice of $c-1$ coloops or choice of $r-c$ remaining elements. In the second case $n-r$ coloops can be used as the elements for "prolonging" the game. The outcome depends exactly on whether $c < (r+1)/2$ or $c > (r+1)/2$. (The first player decides in the case of equality.)

In our opinion, nice characterizations of "winning matroids" for a player can hardly be expected unless F is described by solely numerical parameters.

4. Games on partitional matroids

Let M denote a partitional matroid (see, e.g., [8]), i.e., a pair (S,A), where S ($|S|=n$) is a union of disjoint finite sets $S[1],\dots S[p]$, while A is a p-tuple $(a[1],\dots,a[p])$ of integers, s.t. $0 \le a[i] \le n[i]=|S[i]|$, for $1 \le i \le p$. In other words, M is a direct sum of uniform matroids $U_{n[i],a[i]}$, for $1 \le i \le p$.

Bases of M are $(a[1] + \dots a[p])$-sets, which have $a[l]$ elements in $S[l]$, for $1 \le l \le p$. Hyperplanes of M are $(n-(n[l] - a[l] + 1))$-subsets of S, such that their complements belong to $S[l]$. Circuits of M are $(a[l]+1)$-subsets of $S[l]$ for some l between 1 and p.

$$\text{ACH CIRC: } L = 2 + \sum_{l=1}^{p} (a[l] - 1) + \text{number of free matroids}$$

which are direct summands of M.

Proof. The player who chooses the $(a[l])$-th element from a set $S[l]$ is the loser unless $n[l] = a[l]$. The maximal number of moves which avoid such a situation is equal to the above L-2. #

We say that a set $S[i]$ is *reached, subreached, completed, subcompleted* if the number of already chosen elements from it is not smaller than $a[l]$, $a[l]-1$, $n[l]$, $n[l]-1$, respectively.

ACH BASE: L is equal to some of the values:

$$b[l] = n - (n[l] - a[l])$$

$$c[l,j] = n - (n[l] - a[l]) - (n[j] - a[j]),$$

where $1 < l < j < p$ (if $p=1$, then $b[1] = a[1]$)

Proof. Consider the situation just after reaching the $(p-1)$-th set $S[j]$. If the only unreached $S[l]$ is not subreached, then $L=b[l]$, since the player who chooses the $(a[l]-1)$-th element element from $S[l]$ loses. In the opposite case, the player who makes the $(p-1)$-th reaching is the loser, which means that his last move was forced. This happens only if the sets $S[j]$ and $S[l]$ are subreached, while the other p-2 sets are completed (an immediate consequence is $L = c[l,j]$).

Partial solution: Let $A1 = \{l \mid b[l] \text{ is odd}\}$,

$$A2 = \{l \mid b[l] \text{ is even}\}.$$

Since $n + c[l,j] = b[l] + b[j]$, it follows that, depending on the parity of n, one of the players - denote him by P ("purist") - wins for $L= c[i,j]$, where l and j belong to the same one of the sets $A1$, $A2$, while the other one - denote him by M ("mixer") - wins in the case when l and j belong to the different of the two sets. Further, let $W[P]=\{l \mid P$ wins for $L=b[l]\}$, and similarly define $W[M]$.

Theorem. *If* $\sum_{l\in W[P]} a[l] > \sum_{l\in W[M]} a[l]$,

then the player P wins in ACH BASE (if P is the first, then the equality is also allowed).

Proof. The condition given in the theorem enables P to reach all the sets $S[i]$ with $i \in W[M]$. This implies that $L \in \{b[i], c[i,j]\}$ with some $i, j \in W[P]$, which proves the theorem. #

Remark. If the condition of this theorem is not satisfied, then the solution seems to be very difficult. The player M has not always a winning strategy. For example, if $a[i]=1$ for all i and $|W(M)|>|W(P)|+1$, then the player P can surely win by making two last unreached sets be with i in $W(M)$.

AV BASE: *L is equal to some of the values* $b[i]$.

Proof. None of the players wants to produce the p-th reaching. This implies that the maximal number of "waiting" moves if $b[k]-1$, where $S[k]$ denotes the last unreached set.

Solution:

The first player wins iff

$$\sum_{i \in A2} a[i] \geq \sum_{i \in A1} a[i]$$

(otherwise the second player wins).

Proof. The first player should try to reach all the sets $S[i]$ with $i \in A1$, before the second player reaches the opposite goal. Such strategies lead to the above numerical solution.

AV HYP: *L is equal to* $n-1$ *or to some of the values* $b[i]-1$.

Proof. Consider the situation just after the $(p-1)$-th completion. If the last uncompleted set $S[k]$ is not yet subreached, then obviously $L = b[k]-1$. Otherwise, the $(p-1)$-th completion loses, which implies that it was forced. The only such possibility is that $p-2$ sets $S[i]$ are completed, while the last two are subcompleted only (thus $L=n-1$).

ACH HYP: *The possible values of L are the same as with* AV HYP.

Proof. The first branch is the same as with the previous proof. However, the $(p-1)$-th completion now wins in the second case. Suppose that the number of chosen elements in the last uncompleted set $S[k]$ belongs to $[a[k], n[k]-2]$. This implies that the loser did not play rationally in his previous move. He could either win by making the $(p-1)$-th completion, or "keep the position" by choosing an element from $S[k]$.

Consequence. *The games* ACH HYP *and* AV HYP *can be solved by using the same strategies, but played by opposite players.*

Common solution for ACH HYP and AV HYP:

Let A denote the player who wins (in the considered one of the two games) for $L = n-1$ and let B denote his opponent. Further, let $W[A] = \{i \mid A$ wins for $L = b[i]-1\}$ and similarly define $W[B]$.

Theorem. *If* B *is the second player, then* B *wins iff* $(\exists i \in W[B])\ (a[i]-1 > n-n[i])$. *Almost the same statement holds when* B *is the first, but then the inequality should be weakened (the case of equality is also "winning" for* B*).*

Proof. The winning strategy of B is obvious under the condition above: during the game he should choose the elements outside $S[i]$ only. In this way the $(p-1)$-th completion will happen before $S[i]$ is subreached (thus $L = b[j]-1$ with $j \in W[B]$).

On the other hand, the player A wins if he makes the following goal before the $(p-1)$-th completion:

Subreach all the sets $S[j]$ with $j \in W[B]$.

Namely, after achieving this goal, the only uncompleted set may be subreached in the moment of the $(p-1)$-th completion, causing $L=n-1$, or not, causing $L = b[j]-1$ with $j \in W[A]$.

There remains to show that A can reach his goal provided that the condition given in the theorem is not satisfied.

Let $q[j] = n[j] - (a[j]-1)$ and let $t[j]$ denote the temporary number of remaining elements necessary for subreaching $S[j]$. Each element chosen from $S[j]$ decreases $t[j]$ by 1. The strategy of A may be the following:

Choose an element from some $S[k]$, such that $k \in W[B]$ and

$$t[k] = \max_{j \in W[B]} t[j]$$

Specially, if the only two non-zero t-values are equal to 2, while the corresponding q-values are 1 and val (val > 1) respectively, then A should choose an element from the set corresponding to val.

Thus A will try to "prevent neglecting" any of the sets $S[j]$ with $j \in W[B]$, in order to make the last of them subreached as soon as possible.

Suppose that the first element chosen by A belongs to a set $S[i]$. While (if at all) $t[i]$ is the unique maximum, A will keep choosing elements from $S[i]$. If the condition given in the theorem is not satisfied, then this process will stop without making A to be the loser. Let $t[u]$ and $t[v]$ ($t[u]$ 0 $\geq t[v]$; $u, v \in W[B]$) be the two largest remaining t-values at that moment. It is easy to observe that $t[u] \le t[v]+1$ and that, according to A's strategy, the difference between the two largest t-values can never be greater than 2 (if it is 2, then it is made by B).

In the final stage of the game, $t[u]$ and $t[v]$ become the last two nonzero t-values. A's strategy does not allow some of the sets $S[u]$, $S[v]$ to be subreached before $\max\{t[u], t[v]\}$ becomes < 2.

The final situation $t[u]=2$; $t[v]=0$ is extremely convenient for B. However, a necessary condition for his winning is that $q[v] > 1$, otherwise A makes his goal in the next two moves. It follows that in the compulsory previous position $t[u]= t[v]=2$, A should behave as in the described above special part of his strategy. At last, if $q[u]=q[v]=1$, then there exists a possibility for B to make the $(p-1)$-th completion on the set $S[v]$. However, this would lead to $L = n-1$, which means that A is the winner again.

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Rezime

NEKE IGRE POSTIZANJA I IZBEGAVANJA NA PARTICIJSKIM MATROIDIMA

U radu su rešavane igre postizanja i izbegavanja ciklova, baza i hiperravni na particijskim matroidima. Igra postizanja baze je rešena delimično, dok je za igru izbegavanja cikla dato i rešenje na opštim matroidima. Kod ovih igara dva igrača naizmenično biraju elemente nosača matroida, koji se akumuliraju u neki izabrani skup. Igra je završena nakon poteza kojim se postiže da izabrani skup uključi u sebe neki cikl (bazu, hiperravan) matroida. U igri postizanja je igrač koji povuče poslednji potez - pobednik, dok u igri izbegavanja on gubi. Rešenja na particijskim matroidima se mogu odrediti prvenstveno zahvaljujući činjenici da se ti matroidi mogu kompletno opisati pomoću numeričkih parametara.

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