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ABELIAN THEOREMS FOR THE OBRECHKOFF INTEGRAL TRANSFORM

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Abstract

In 1958 Obrechkoff [1] introduced a generalization of the integral transforms of Laplace and Meijer. In the same paper some differential and asymptotic properties of its kernel function were investigated and a real inversion formula of the Post- Widder type was found. Later on, Dimovski [2], [3] proposed a modification of this transform, usually referred to as the Obrechkoff integral transform. As was shown in [2]-[5], it can be used as a basis of an operational calculus for the most general Bessel type differential operator of an arbitrary order. It has turned out that a number of Bessel type integral transformations proposed by different authors are quite special cases of the Obrechkoff transform. Here, we shall propose Abelian theorems for this transform, that is "initial (final) value" theorems relating the initial (final) value of an original to the final (initial) value of its transformation.

1. Some preliminary results on the Obrechkoff integral transform

Definition 1. *Let $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_m$ be an arbitrary sequence od m real numbers, $\beta > 0$ be arbitrary too and*

$$(1)\quad K(s) = \int_0^\infty \ldots \int_0^\infty exp(-u_1 - u_2 - \ldots - u_{m-1} - \frac{s}{u_1 \ldots u_{m-1}}) \prod_{k=1}^{m-1} u_k^{\gamma_k - \gamma_m - 1} \quad du_1 \ldots du_{m-1}.$$

The integral transform

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$$(2)\qquad \mathcal{O}(s) = \mathcal{O}\left\{f(x);s\right\} = \beta\int_0^\infty K\left[(sx)^\beta\right] x^{\beta(\gamma_m+1)-1} f(x)dx,$$

defined for functions on $0\le x<\infty$, *is said to be the* **Obrechkoff integral transform.**

Denote by $L_\alpha(C_\alpha)$ the space of Lebesgue integrable (continuous) functions on $(0,\infty)$ with a prescribed power growth at $x=0$:

$$(3)\qquad L_\alpha = \left\{f(x)=O(x^\rho),\ \rho>\alpha \ \text{ as } \ x\to 0;\ f\in L(0,\infty)\right\},$$

$$(4)\qquad C_\alpha = \left\{f(x)=x^\rho\tilde{f}(x) \ \text{ with } \ \rho>\alpha,\ \tilde{f}\in C[0,\infty)\right\}.$$

The subspace of L_α consisting of the functions having an exponential growth at infinity:

$$(5)\qquad \Omega := L^{exp}_{\alpha,\frac{\beta}{m}} = L_\alpha \cap \left\{f(x)=O\left[\lambda x^{\frac{\beta}{m}}\right],\ \lambda\in\mathbb{R} \ \text{ as } \ x\to\infty\right\}$$

is said to be the space of *Obrechkoff transformable functions. Denote by* $C^{exp}_{\alpha,\frac{\beta}{m}}$ the more explicitly written subspace of Ω :

$$(6)\qquad C^{exp}_{\alpha,\frac{\beta}{m}} = C_\alpha \cap \left\{f(x)=O\left[\lambda x^{\frac{\beta}{m}}\right],\ \lambda\in\mathbb{R} \ \text{ as } \ x\to\infty\right\}.$$

For each function $f\in\Omega$ $\left[f\in C^{exp}_{\alpha,\frac{\beta}{m}}\right]$ there exists a well-defined Obrechkoff transform $\mathcal{O}(s)$ which is an analytic function in the truncated angle domain $D=\left\{\operatorname{Re} z>\lambda\right\}\cap\left\{|\arg z|<\frac{\pi m}{2\beta}\right\}$.

Dimovski [2] - [5] used transform (2) as the basis of an operational calculus for the *Bessel type differential operator of arbitrary order* $m>1$:

$$(7)\qquad B = x^{-\beta}\prod_{k=1}^{m}\left(x\frac{d}{dx}+\beta\gamma_k\right) = x^{-\beta}Q_m\left(x\frac{d}{dx}\right),\quad 0<x<\infty$$

($Q_m(\mu)$ is a m-th degree polynomial with zeros $\mu_k=-\beta\gamma_k$, $k=1,\dots,m$, having also the representation

(8) $$B = x^{\alpha_0}\frac{d}{dx}x^{\alpha_1}\cdots\frac{d}{dx}x^{\alpha_{m-1}}\frac{d}{dx}x^{\alpha_m}$$

with $\alpha_0=-\beta-\beta\gamma_1+1$; $\alpha_k=\beta\gamma_k-\beta\gamma_{k+1}+1$, $k=1,\dots,m-1$; $\alpha_m=\beta\gamma_m$ and $\beta=m-(\alpha_0+\dots+\alpha_m)>0$. It is worth mentioning here following *special cases* of the Obrechkoff transform, proposed by different authors and used by them in developing operational calculi for Bessel type operators of form (7):

i) the well-known *Meijer transform*

(9) $$\mathcal{K}_\nu\left\{f(x);s\right\}=\int_0^\infty\sqrt{sx}\,K_\nu(sx)\,f(x)dx,$$

related to the second order Bessel differential operator

$$B_\nu=\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{\nu^2}{x^2}=x^{-2}\left(x\frac{d}{dx}-\nu\right)\left(x\frac{d}{dx}+\nu\right),\quad \nu\ge 0,$$

with $m=\beta=2$, $\gamma_1=-\frac{\nu}{2}$, $\gamma_2=-\frac{\gamma}{2}$.

ii) the integral transformation used by *Meller* [6] as a transform basis of an operational calculus for the Bessel type differential operator $B_\alpha=x^{-\alpha}\frac{d}{dx}x^{\alpha+1}\frac{d}{dx}$, $-1<\alpha<1$. When $\alpha=0$ we get *Ditkin*'s integral transformation ([7]):

$$\mathcal{K}\left\{f(x);s\right\}=2s\int_0^\infty K_0\left(2\sqrt{sx}\right)f(x)dx,$$

corresponding to the operator $B_0=\frac{d}{dx}x\frac{d}{dx}$.

iii) a more general integral transformation related to the differential operator $B^{(m)}=\frac{d}{dx}x\frac{d}{dx}\cdots x\frac{d}{dx}=\frac{1}{x}\left[x\frac{d}{dx}\right]^m$ of order $m>1$, was proposed simultaneously by *Ditkin* and *Prudnikov* [8] and by *Botashev* [9]:

$$\mathcal{O}\left\{f(x);s\right\}=2\int_0^\infty E_{0m}(sx)f(x)dx,$$

where

$$E_{0m}(s)=\frac{1}{4\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma^m(\sigma)}{\{(-1)^m s\}^\sigma}\,d\sigma.$$

iv) In a number of papers *Krätzel* considered the Bessel type operator of order $m>1$:

$$B = D_{m,\nu} = \frac{d}{dx} x^{\frac{1}{m}-\nu}\left(x^{1-\frac{1}{m}}\frac{d}{dx}\right)^{m-1} x^{\nu+1-\frac{2}{m}}.$$

In [10] he used the following generalization of the Laplace and Meijer transforms:

$$\mathcal{L}_{\nu}^{(m)}\left\{f(x);s\right\} = (2\pi)^{\frac{m-1}{2}}\sqrt{m}\, s^{\nu}\int_0^{\infty}\lambda_{\nu}^{(m)}\left[(sx)^{\frac{1}{m}}\right]x^{\nu}f(x)dx$$

with a kernel-function of the form

$$\lambda_{\nu}^{(m)}(s) = \int_1^{\infty}\frac{(\tau^{m}-1)^{\nu-\frac{1}{m}}}{\Gamma(\nu-\frac{1}{m}+1)}\exp(-ms\tau)d\tau.$$

This transformation can be obtained from the Obrechkoff transform (2) by a specialization of the parameters, namely for

$$\gamma_1 = \frac{2}{m}-\nu-1;\quad \gamma_k = \frac{k}{m},\ k=2,\ldots,m-1;\quad \gamma_m = \nu+1-\frac{2}{m};\quad \beta=1.$$

Dimovski [2], [3] found some operational properties and a *convolution* of the Obrechkoff transform. Further, in [11] a *real inversion formula* for the modified transform (2) is proposed and the following *differential property*, on which the corresponding operational calculus is based, was established:

$$O\{Bf(x);s\} = \beta^{m}s^{\beta}\,O\{f(x);s\}$$

(10)

$$-\sum_{i=1}^{m}\left[\beta^{i}s^{\beta(\gamma_i-\gamma_m)}\prod_{j=1}^{i-1}\Gamma(\gamma_j-\gamma_i+1)\prod_{j=i+1}^{m}\Gamma(\gamma_j-\gamma_i)\right]\lim_{x\to 0}\left[x^{\beta\gamma_i}\prod_{j=i+1}^{m}\left(x\frac{d}{dx}+\beta\gamma_j\right)\right]f(x).$$

A number of *complex inversion formulas* for the Obrechkoff transform are obtained in [11], [12], [13]. By specialization of the parameters $\beta > 0$, $\gamma_1\le\ldots\le\gamma_m$ the results of [2]-[5], [11]-[14] turn into corresponding operational properties, inversion formulas etc. for the aforesaid transformations i)- iv).

It is interesting to note ([12], [14]) that the kernelfunction (1) of the Obrechkoff transform is a special case of *Meijer's G-function* (see e.g. [15], p. 203):

(11) $$G^{m,n}_{p,q}\left[\sigma\left|\begin{matrix}(a_j)_{j=1}^{p}\\(b_k)_{k=1}^{q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\prod\limits_{k=1}^{m}\Gamma(b_k+\xi)\prod\limits_{j=1}^{n}\Gamma(1-a_j+\xi)}{\prod\limits_{k=m+1}^{q}\Gamma(1-b_k+\xi)\prod\limits_{j=n+1}^{p}\Gamma(a_j+\xi)}\sigma^{\xi}d\xi.$$

This allows to give a new representation of (2) by means of a simple integral only, and to define it as a special case of the so-called *G-transforms*, viz. ([13], [16]):

Definition 1. *The G-transform of the form*

(12) $$O\{f(x);s\} = \beta s^{-\beta(\gamma_m+1)+1}\int_0^{\infty} G^{m,0}_{0,m}\left[(sx)^{\beta}\left|(\gamma_k-\frac{1}{\beta}+1)_{k=1}^{m}\right.\right]f(x)dx$$

is said to be the Obrechkoff integral transform corresponding to the hyper-Bessel differential operator $B = x^{-\beta}\prod\limits_{k=1}^{m}\left(x\frac{d}{dx}+\beta\gamma_k\right)$ *with* $\gamma_1\le\gamma_2\le\ldots\le\gamma_m$.

As it is show in [13], [16], this new representation of (2) simplifies essentially all previous considerations concerning this transformation.

There exists also a *relationship between the Obrechkoff transform and the usual Laplace transform*

(13) $$\mathfrak{L}\{f(x);s\} = \int_0^{\infty} e^{-sx} f(x)dx.$$

It is given by the equality ([12], [14])

(14) $$O\left\{f(x);\left[\frac{s}{m}\right]^{\frac{m}{\beta}}\right\} = \left[2\pi\right]^{\frac{m-1}{2}}\sqrt{m}\ \mathfrak{L}\{\varphi f(x);s\},\ f\in\Omega,$$

where $\varphi f(x)$ denotes the *generalized Sonine transformation* corresponding to the hyper-Bessel differential operator B, proposed by Dimovski [3], [5]:

$$\varphi f(x)=x^{m(\gamma_m+1)-1}\int_0^1\ldots\int_0^1\left[\prod_{k=1}^{m-1}\frac{\left[1-\sigma_k\right]^{\gamma_m-\gamma_k+\frac{k}{m}-1}}{\Gamma\left[\gamma_m-\gamma_k+\frac{k}{m}\right]}\sigma_k^{\gamma_k}\right]f\left[x^{\frac{m}{\beta}}(\sigma_1\ldots\sigma_{m-1})^{\frac{1}{\beta}}\right]d\sigma_1\ldots d\sigma_{m-1}.$$

(15)

This relation has an important meaning as far as it allows the transferring of the well-known results for the Laplace transform into corresponding ones for the Obrechkoff transform. Especially, here *we shall demonstrate its* *advantages in deriving Abelian type theorems for* (2). To this end some properties of the Sonine transformation (15) discussed in view of the theory of the generalized fractional integration operators will be needed.

2. A brief sketch of the generalized fractional calculus

In [16], [17] a generalized fractional calculus is developed dealing with operators of the integration and differentiation of fractional multiorder. These operators, being multidimensional compositions of classical fractional integrals or derivatives, have simple integral representations using another case of Meijer's G-function (11) as a kernel. Here we shall need only the definition of the generalized fractional integrals and a few of their properties used in our considerations. For the full exposition of this theory and many special cases one can see [16], partially- [17], [18]. Various applications in the theory of special functions, in solving quite general classes of differential equations and dual integral equations, in operational calculus and integral transforms etc. are proposed in other papers by Kiryakova.

Definition 2. *Let $m\geq 1$ be an integer, $\beta>0$, the "weights" $\gamma_1, \gamma_2, \dots, \gamma_m$ be arbitrary real numbers and $\delta_1\geq 0, \delta_2\geq 0, \dots, \delta_m\geq 0$ be the components of the "multiorder of integration" $\delta=(\delta_1,\dots,\delta_m)$. Define the integral operators*

$$(16)\qquad \mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = \int_0^1 G_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}(\gamma_k+\delta_k)_{k=1}^m\\(\gamma_k)_{k=1}^m\end{matrix}\right.\right] f\left(x\sigma^{\frac{1}{\beta}}\right)d\sigma,$$

depending on these $(2m+2)$ parameters. Then every operator of the form

$$(17)\qquad \mathcal{R} f(x) = x^{\beta\delta_0}\,\mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) \qquad \text{with some } \delta_0\geq 0,$$

is said to be a **generalized (m- dimensional) operator of the fractional integration** *of the Riemann-Liouville type, or briefly: a generalized R.-L. fractional integral.*

It is shown that the operators $\mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)}$ are linear invertible mappings of the spaces C_α defined by (4) with $\alpha \geq \max_k\left[-\beta\,(\gamma_k+1)\right]$, into themselves, i.e. $\mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} : C_\alpha \to C_\alpha$. The same can be said for spaces (3) of Lebesgue integrable function on $(0,\infty)$ having a prescribed power growth at $x=0$, namely $\mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} : L_\alpha \to L_\alpha$, $\alpha \geq \max_k \left[-\beta(\gamma_k+1)\right]$.

This fact becomes more explicable if we take into account that operators (16) preserve the power functions up to a constant multiplier:

$$(18)\qquad \mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)}\{x^\rho\} = c_\rho x^\rho = x^\rho \prod_{k=1}^{m} \frac{\Gamma(\gamma_k+\frac{\rho}{\beta}+1)}{\Gamma(\gamma_k+\delta_k+\frac{\rho}{\beta}+1)}, \quad \rho > \alpha ,$$

and therefore they preserve the asymptotic behaviour of the functions $f(x)$ which are $O(x^\rho)$ near to the initial point $x=0$:

$$(18')\qquad |f(x)| \leq Ax^\rho, \ x \to 0 \quad \Rightarrow \left|\mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} f(x)\right| \leq Ac_\rho x^\rho, \ x \to 0.$$

Other rules which will be useful here are:

$$(19)\qquad \mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} x^{\beta\lambda} f(x) = x^{\beta\lambda}\, \mathcal{R}_{\beta,m}^{(\gamma_k+\lambda)(\delta_k)} f(x),$$

$$(20)\qquad \left\{\mathcal{R}_{\beta,m}^{(\gamma_k),(\delta_k)} f\right\}^{(j)}(0) = f^{(j)}(0) \prod_{k=1}^{m} \frac{\Gamma(\gamma_k+\frac{j}{\beta}+1)}{\Gamma(\gamma_k+\delta_k+\frac{j}{\beta}+1)}, \quad j=0,1,2,\ldots .$$

It is worth saying also that all the known results for the Riemann-Liouville and Erdélyi-Kober fractional integrals and for the classical fractional derivatives have their counterparts concerning the generalized fractional integrals (16), (17) and the generalized fractional derivatives corresponding to them (see e. g. [16], [17], [18]).

What will be used now is *another representation of the generalized Sonine transformation (15)* by means of a simple integral involving Meijer's $G_{m-1,m-1}^{m-1,0}$- function as kernel, that is, *as a generalized $(m-1)$-dimensional fractional integral* (see [13], [16], [18]):

$$(21)\qquad (\varphi f)\left(x^{\frac{\beta}{m}}\right) = x^{\beta(\gamma_m+1)-\frac{\beta}{m}}\, \mathcal{R}_{\beta,m-1}^{(\gamma_k),(\gamma_m-\gamma_k+\frac{k}{m})} f(x).$$

3. Initial value theorem for the Obrechkoff transform

The knowledge of Abelian theorems for a given integral transform is of considerable importance for its use in solving initial and boundary value problems arising in mathematical physics. For the Laplace transform (13) several different variants of the initial and final value theorems (Abelian theorems) are known (see [19], [20]). One of them can be stated as follows.

Lemma. Let $f(x)$ be a Lebesgue integrable function on $(0,\infty)$ which is $O(e^{\lambda x})$ with some positive number λ as $x \to \infty$. Assume that there exists the limit

(22) $$\lim_{x\to+0} x^{-\rho} f(x) = f_0$$

and consider the Laplace transform

$$\mathcal{L}(s) = \mathcal{L}\{f(x);s\} = \int_0^\infty e^{-sx} f(x)\,dx, \quad s > \lambda$$

Then,

(23) $$\lim_{s\to+\infty} s^{\rho+1}\mathcal{L}(s) = f_0\Gamma(\rho+1),$$

or

(24) $$f(x) = O(x^{\rho}),\ x \to 0 \Rightarrow \mathcal{L}(s) = O(s^{-\rho-1}),\ s \to \infty.$$

Remark 1. The conditions required for the original function $f(x)$ are fulfilled for instance in the subspace of continuous function of the form (6) with $\alpha=-1$, $m=\beta=1$:

(25) $$C^{exp}_{-1,1} = \left\{ f(x)=x^{\rho}\tilde{f}(x),\ \rho>-1,\ \tilde{f} \in C[0,\infty);\ f(x)=O(e^{\lambda x}),\ x \to \infty \right\}.$$

In this case

(22') $$\lim_{x\to+0} x^{-\rho} f(x) = \lim_{x\to+0} \tilde{f}(x) = \tilde{f}(0) := f_0$$

exists and yields the existance of the limit

(23') $$\lim_{s\to+\infty} s^{\rho+1}\mathcal{L}(s) = \tilde{f}(0)\ \Gamma(\rho+1).$$

Initial and final value theorems for the Hankel transform and for the Meijer transform (9) are proposed by *Zemanian* [21]. For the general G-transformation of *Narain* [22] and *Kesarwani* [23] analogous and quite general results are obtained by *Pathak* [24]. In view of the new Definition 1' one can find Abelian theorems for the Obrechkoff transform (2) by a suitable specialization of Pathak's results and by a specification of the list of conditions imposed on the original function $f(x)$ there. Independent of this, we shall demonstrate now that the relationship (14) between the *Obrechkoff and Laplace integral transforms* and the properties of the generalized fractional integral (21) *can be used to this end too.*

Theorem 1. *Let $f \in \Omega$ be an Obrechkoff transformable function and $\rho > \alpha = \max\limits_{1\le k\le m}\left[-\beta(\gamma_k+1)\right] = -\beta(\gamma_1+1)$. Assume there exists a limit* (22):

$$\lim_{x\to+0} x^{-\rho}f(x) = f_0$$

and denote by

$$\mathcal{O}(s) = O\left\{f(x);s\right\},\ s>\lambda,$$

the Obrechkoff transformation (2). Then there exists a limit

$$\lim_{s\to\infty} s^{\rho+\beta(\gamma_m+1)}\mathcal{O}(s) = f_0\prod_{k=1}^{m}\Gamma\left(\gamma_k+\frac{\rho}{\beta}+1\right). \tag{26}$$

In the case of a continuous function $f(x)$ on $(0,\infty)$ which belongs to the subspace $C^{\exp}_{\alpha,\frac{\beta}{m}}\subset\Omega$, this assertion has the following more explicit form.

Theorem 1'. *Suppose that $f \in C^{\exp}_{\alpha,\frac{\beta}{m}}$, i.e. it has the form*

$$f(x) = x^{\rho}\tilde{f}(x) \quad \textit{with some } \rho > \alpha \textit{ and } \tilde{f} \in C(0,\infty), \tag{27}$$

and is $O\left(\exp\lambda x^{\frac{\beta}{m}}\right)$ as $x\to\infty$ with some real positive λ. Then,

$$\lim_{s\to\infty} s^{\rho+\beta(\gamma_m+1)}\mathcal{O}(s) = \tilde{f}(0)\prod_{k=1}^{m}\Gamma\left(\gamma_k+\frac{\rho}{\beta}+1\right). \tag{26'}$$

Proof. For $f(x) = x^{\rho}\tilde{f}(x)$ with $\rho > \alpha$, $\tilde{f} \in C[0,\infty)$, condition (22') is satisfied. In [3], [16] it is proved that the Sonine transformation φ maps the space C_{α}, $\alpha \geq -\beta(\gamma_1+1)$ into the space C_{-1} where Laplace transform appearing on the right-hand side of (14) can be considered. To describe more precisely the asymptotic behaviour of $\varphi f(x)$ as $x \to 0$, we shall use the properties of the generalized fractional integral (21) mentioned in 2. According to (19)

$$\mathcal{R}_{\beta,m-1}^{(\gamma_k),(\gamma_m-\gamma_k+\frac{k}{m})} f(x) = \mathcal{R}_{\beta,m-1}^{(\gamma_k),(\gamma_m-\gamma_k+\frac{k}{m})} x^{\rho}\tilde{f}(x) = x^{\rho}\, \mathcal{R}_{\beta,m-1}^{(\gamma_k+\frac{\rho}{\beta}),(\gamma_m-\gamma_k+\frac{k}{m})} \tilde{f}(x),$$

that is

$$(\varphi f)\left(x^{\frac{\beta}{m}}\right) = x^{\beta(\gamma_m+1)-\frac{\beta}{m}}\, \mathcal{R}_{\beta,m-1}^{(\gamma_k),(\gamma_m-\gamma_k+\frac{k}{m})} \tilde{f}(x) = x^{\beta(\gamma_m+1)-\frac{\beta}{m}+\rho}\, \hat{f}(x),$$

where $\hat{f}(x) = \mathcal{R}_{\beta,m-1}^{(\gamma_k+\frac{\rho}{\beta}),(\gamma_m-\gamma_k+\frac{k}{m})} \tilde{f}(x)$ is denoted. So

$$\varphi f(x) = x^{m(\gamma_m+1)-1+\frac{m\rho}{\beta}}\, \hat{f}\left(x^{\frac{m}{\beta}}\right)$$

and hence

$$\lim_{x\to 0} x^{-m(\gamma_m+\frac{\rho}{\beta}+1)+1}\varphi f(x) = \lim_{x\to 0} \hat{f}\left(x^{\frac{m}{\beta}}\right) = \hat{f}(0).$$

On the other hand,

$$\hat{f}(0) = \left[\mathcal{R}_{\beta,m-1}^{(\gamma_k+\frac{\rho}{\beta}),(\gamma_m-\gamma_k+\frac{k}{m})}\tilde{f}\right](0) = \tilde{f}(0)\prod_{k=1}^{m-1}\frac{\Gamma(\gamma_k+\frac{\rho}{\beta}+1)}{\Gamma(\gamma_m+\frac{k}{m}+\frac{\rho}{\beta}+1)},$$

by virtue of property (20) for $j=0$. In this manner we get that condition (22') implies the condition

$$\lim_{x\to 0} x^{-m(\gamma_m+\frac{\rho}{\beta}+1)+1}\varphi f(x) = \tilde{f}(0)\prod_{k=1}^{m-1}\frac{\Gamma(\gamma_k+\frac{\rho}{\beta}+1)}{\Gamma(\gamma_m+\frac{k}{m}+\frac{\rho}{\beta}+1)} \tag{28}$$

where $\varphi f(x)$ is the generalized Sonine transformation (15). There remains to apply the initial value Lemma for the Laplace transform on the right -hand side of equality (14). Condition (28) satisfied by $\varphi f(x)$ yields that limit (23') exists when ρ is substituted by $(m(\gamma_m+\frac{\rho}{\beta}+1)-1)$ and f_0 is substituted by $\tilde{f}(0)\prod_{k=1}^{m-1}\Gamma(\gamma_k+\frac{\rho}{\beta}+1)$, i.e.

$$\lim_{s\to\infty} s^{m(\gamma_m+\frac{\rho}{\beta}+1)}\mathcal{L}\{\varphi f(x);s\} = \tilde{f}(0)\Gamma(m(\gamma_m+\frac{\rho}{\beta}+1))\prod_{k=1}^{m-1}\frac{\Gamma(\gamma_k+\frac{\rho}{\beta}+1)}{\Gamma(\gamma_m+\frac{k}{m}+\frac{\rho}{\beta}+1)}.$$

Now the Gauss-Legendre multiplication formula for Γ- functions ([15], p. 18) has to be applied:

$$\Gamma(m(\gamma_m+\frac{\rho}{\beta}+1)) = (2\pi)^{-\frac{m-1}{2}} m^{-\frac{1}{2}} m^{m(\gamma_m+\frac{\rho}{\beta}+1)}\Gamma(\gamma_m+\frac{\rho}{\beta}+1)\prod_{k=1}^{m-1}\Gamma(\gamma_m+\frac{k}{m}+\frac{\rho}{\beta}+1).$$

Hence, we get that

$$\text{(29)}\quad \lim_{s\to\infty} s^{m(\gamma_m+\frac{\rho}{\beta}+1)}\mathcal{L}\{\varphi f(x);s\} = \tilde{f}(0)(2\pi)^{-\frac{m-1}{2}} m^{-\frac{1}{2}} m^{m(\gamma_m+\frac{\rho}{\beta}+1)}\prod_{k=1}^{m}\Gamma(\gamma_m+\frac{\rho}{\beta}+1),$$

or

$$\lim_{s\to\infty}\mathcal{F}(s) = \tilde{f}(0)\prod_{k=1}^{m}\Gamma(\gamma_k+\frac{\rho}{\beta}+1),$$

where

$$\mathcal{F}(s) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}}\left(\frac{s}{m}\right)^{m(\gamma_m+\frac{\rho}{m}+1)}\mathcal{L}\{\varphi f(x);s\}$$

is denoted. According to (14)

$$\mathcal{F}(s) = \left[\left(\frac{s}{m}\right)^{\frac{m}{\beta}}\right]^{\rho+\beta(\gamma_m+1)} O\left\{f(x);\left(\frac{s}{m}\right)^{\frac{m}{\beta}}\right\},$$

and as

$$\lim_{s\to\infty}\mathcal{F}(s) = \lim_{\substack{m\sigma^{\frac{\beta}{m}}\to\infty}}\mathcal{F}\left(m\sigma^{\frac{\beta}{m}}\right) = \lim_{\sigma\to\infty}\sigma^{\rho+\beta(\gamma_m+1)} O\{f(x);\sigma\},$$

we obtain the limit (26'). Let us note that the condition $f(x)=O\left(e^{\lambda x^{\frac{\beta}{m}}}\right)$ as $x\to\infty$ is needed to ensure the convergence of the Obrechkoff integral (2) for

$s>\lambda$, and, therefore, the existance of the Obrechkoff image $\mathcal{O}(s)$ in the same way as the condition $f(x)=O\left(e^{\lambda x}\right)$, $x\to\infty$ ensures the convergence of the Laplace integral (13) for $s>\lambda$. This fact is a corollary of Obrechkoff's results [1] determining the asymptotic behaviour of the kernel-function (1) as $s\to\infty$:

$$(30)\qquad K(s)\sim(2\pi)^{\frac{m-1}{2}}m^{-\frac{1}{2}}s^{\left[\frac{m-1}{2}+\sum_{1}^{m-1}(\gamma_k-\gamma_m-1)\right]/m}\exp\left(-ms^{\frac{1}{m}}\right).$$

The same asymptotic formula can be derived directly from the theory of G-functions taking into account Definition 1'. The proof is over.

Remark 2. To prove Theorem 1 in the more general case of Lebesgue integrable function $f\in\Omega$, we have to make only slight modifications. It is sufficient to replace condition (22') by the supposition that the limit $\lim_{x\to+0}\tilde{f}(x)=f_0$ exists. This will imply the existence of limit (26).

Remark 3. The result of *Theorem 1 follows also from the general consideration of Pathak* [24] if one substitutes the parameters used there by the following ones:

$$m\mapsto m,\ p\mapsto 0,\ q\mapsto 0,\ n\mapsto 0,\ \gamma\mapsto\frac{\beta}{2},\ c_k=\gamma_k-\frac{1}{2\beta}+\frac{1}{2},$$
$$\beta\mapsto\beta(\gamma_1+1)-1,\ \Delta\mapsto\frac{m}{2}\left(1-\frac{1}{\beta}\right)+\sum_1^m\gamma_k,\ \rho\mapsto m,\ \eta-\frac{1}{2}\mapsto-\rho.$$

Only the following conditions from the statement of Th. 1, [24] have to be retained: A) $(\gamma>0)\mapsto(\beta>0)$; B) iii): $(0\le m-1,\ p=0,\ n=0)\mapsto(m\ge 1)$, $(\eta<\beta+\frac{3}{2})\mapsto(\rho>\alpha=-\beta(\gamma_1+1))$; C) $f(x)$ is a measurable function on $(0,\infty)$ for which integral (2) exists. Then, the existence of a number $\alpha\mapsto f_0$ suchthat $\lim_{x\to+0}x^{-\rho}f(x)=f_0$ yields that (26) in our notations is fulfilled.

4. Final value theorem for the Obrechkoff transform

Now we shall dispose with tree different manners of obtaining a resul't analogous to the final value theorems for the Laplace transform ([19]) and for the Meijer transform ([21]). First, we can prove that $\lim_{x\to+\infty}x^{-\rho}f(x)=f_\infty$ implies that $\lim_{s\to 0}s^{\rho+\beta(\gamma_m+1)}\mathcal{O}(s)=f_\infty\prod_{k=1}^{m}\Gamma(\gamma_k+\frac{\rho}{\beta}+1)$ directly, using the pattern of [19], [21], [24] and Definition 1'. The second way is to derive this result from Pathak's Theorem 2, [24] by the substitutions and specifications mentioned in Remark 3. The last but not least approach is to use the generalized Sonine transformation $\varphi f(x)$ as this is done in 3. One way or another, the following assertion can be established.

Theorem 2. *Let $f \in \Omega$ be an Obrechkoff transformable function and $\rho>\alpha = -\beta(\gamma_1+1)$. The existance of a number f_∞ such that*

$$\lim_{x\to\infty} x^{-\rho} f(x) = f_\infty \tag{31}$$

implies the limit equality

$$\lim_{s\to 0} s^{\rho+\beta(\gamma_m+1)} \mathcal{O}(s) = f_\infty \prod_{k=1}^{m} \Gamma(\gamma_k + \frac{\rho}{\beta} + 1) \tag{32}$$

5. **Example.** As corollaries of Theorem 1, 2, initial and final value theorems for the integral transformations i)-iv) can be obtained. Here we shall demonstrate this only *for the case of the classical Meijer transform* (9). So, by taking

$$m=\beta=2,\ \gamma_1=-\frac{\nu}{2},\ \gamma_2=\frac{\nu}{2}\ (0\le\nu\le\frac{1}{2}),\ \alpha=-\beta(\gamma_1+1)=\nu-2,$$

we receive *the following results of Zemanian* [21], as

$$G^{2,0}_{0,2}\left[\left(\frac{s}{2}\right)^2 \middle| -\frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+\frac{1}{2}\right] = sK_\nu(s)$$

([15], p. 211) and the Obrechkoff transform (12) turns into a modification of the Meijer transform (9).

Corollary 1. ([21], Lemma 3.) Let the numbers ν, ρ be so restricted that $0\le\nu\le\frac{1}{2}$ and $\rho>\nu-2$. Let $f(x)$ be a mesurable function on $(0,\infty)$ such that, for some positive number λ, $f(x)e^{-\lambda x}$ is a Lebesgue integrable on every interval of the form $X<x<\infty$ $(X>0)$. Assume that there exists a number f_0 such that

$$\lim_{x\to 0} x^{-\rho+\frac{1}{2}} f(x) = f_0 \tag{33}$$

Finally, set

$$\mathcal{F}(s) = \int_0^\infty \sqrt{sx}\, K_\nu(sx) f(x)\,dx, \quad s > \lambda. \tag{34}$$

Then,

$$\lim_{s\to+\infty} \mathcal{F}(s) s^{\rho+\frac{1}{2}} = f_0 \mathcal{G}(\nu,\rho), \tag{35}$$

where

$$\mathcal{G}(\nu,\rho) = 2^{\rho-1}\, \Gamma\left(\frac{\rho}{2} + \frac{1}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{\rho}{2} + \frac{1}{2} - \frac{\nu}{2}\right). \tag{36}$$

Corollary 2. ([21], Lemma 7) Let $0 \le \nu \le \frac{1}{2}$ and $\rho > -\frac{1}{2}$. Let $f(x)$ be a measurable function on $(0,\infty)$ satisfying the conditions:

i) $x^{-\nu+\frac{1}{2}} f(x)$ is Lebesgue integrable for $\nu \ne 0$ on every interval of the form $0 < x < X$ $(X < \infty)$, and

ii) there exists a number f_∞ such that

$$\lim_{x\to+\infty} x^{-\rho+\frac{1}{2}} f(x) = f_\infty. \tag{37}$$

Then,

$$\lim_{s\to+0} \mathcal{F}(s) s^{\rho+\frac{1}{2}} = f_\infty \mathcal{G}(\nu,\rho), \tag{38}$$

where $\mathcal{F}(s)$ is defined by (34) for each $s>0$ and $\mathcal{G}(\nu,\rho)$ is the same number as in (36).

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Rezime

ABELOVE TEOREME ZA INTEGRALNU TRANSFORMACIJU OBRECHKOFFA

Koristeci rezultat Pathaka [24] dokazane su Abelove teoreme za integralnu transformaciju Obrechkoffa koja je uopštenje integralne transformacije Laplacea i Meijera.

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