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WEAK CONGRUENCES OF A LATTICE

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Abstract

$C_w(L)$ is a set of all the weak congruences on a given lattice L, i.e. a set of all the congruences on all the sublattices of L.

It is proved here that: a) the lattice $(C_w(L),\le)$ is modular if and only if L is a two-element chain; and b) if L is a bounded lattice (with two disjoint constants 0 and 1) then $(C_w(L),\le)$ is complemented if and only if its lattice of sublattices is complemented. (For lattice L without constants, $(C_w(L),\le)$ is never complemented).

0. Let $A = (A,F)$ be an algebra, and $K \subseteq A$ the set of its constants. Then ([7] *a weak congruence relation on* A is a symmetric, transitive and compatible relation ρ on A, satisfying the weak reflexivity: if $c \in K$, then $c\rho c$.

For an algebra A, denote by

$S(A)$ – the set of its subalgebras;

$C(A)$ – the set of all the congruences on A;

$C_w(A)$– the set of all the weak congruences on A;

$C(B)$ – the set of all the congruences on $B \in S(A)$;

It is obvious that $C_w(A)$ is the set of all the congruences on all the subalgebras of A.

It was proved in [7] that

(I) $(C(A),\le)$ is a sublattice, and $(S(A),\le)$ a retract in $(C_w(A),\le)$. (Therefore, we identify here the subalgebras of A with the corresponding diagonal relations in $C_w(A)$).

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$\mathcal{A}$ is said to have *the congruence intersection property* (CIP), ([7]) if for all $\rho, \theta \in C_w(\mathcal{A})$,

$$(\rho \wedge \theta)_{\Delta} = \rho_{\Delta} \wedge \theta_{\Delta} \;,$$

where

$$\rho_{\Delta} = \cap \, (\alpha \in C(\mathcal{A}) \,|\, \rho \le \alpha)$$

If d_{ρ} is a diagonal of $\rho \in C_w(A)$, i.e. $d_{\rho} = \rho \wedge \Delta$, where $\Delta = \{(x,x) \,|\, x \in A\}$, then $\rho_{\Delta} = \rho \vee \Delta$, and the CIP thus expresses the distributivity ([4]) of Δ in $(C_w(A), \le)$.

$\mathcal{A}$ is said to have *the congruence extension property* (CEP), ([5]), if every congruence on any subalgebra of $\mathcal{A}$ is a restriction of a congruence on $\mathcal{A}$.

It was proved in [7] that (II) $\mathcal{A}$ has a modular lattice of weak congruences if and only if it has the modular lattices of subalgebras and congruences, and it satisfies the CEP, and CIP.

Some algebras having a complemented lattice of weak congruences including the case when this lattice is Boolean were characterized in [6].

1. Consider now the case when $\mathcal{A}$ is a lattice $(L, \wedge, \vee)$ denoted by L. The following propositions are well known.

Proposition 1.1. [1] *$(S(L), \le)$ is a modular lattice if and only if L is a chain.* □

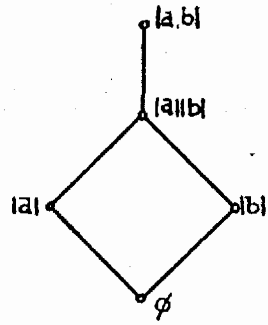
Propisition 1.2. [4]. *For every lattice L, $(C(L), \le)$ is a distributive lattice.* □

Proposition 1.3. [5]. *If L is a distributive lattice then it has the CEP.* □

Thus, to discuss the modularity of the lattice $(C_w(L), \le)$, we can restrict our attention to the case when L is a chain.

Proposition 1.4. *The chain L satisfies the CIP if and only if it has two elements.*

Proof. Let L be a two-element chain $(\{a,b\}, \wedge, \vee)$, $a < b$. Its lattice of weak congruences is represented below. ($|a,b|$ stands for a partition determined by L^2 etc.) L obviously satisfies the CIP. (The same occurs in the case when at least one of those elements is constant in L i.e. when $|K| = 1$, or $|K| = 2$. Now, let L be a chain with more than two elements, and let $L_1 = ((a] \cup [b), \leq)$, where $a, b \in L$, $a < b$, $(a]$ and $[b)$ are the ideal and the filter generated by those two elements. L_1 is obviously a subchain of L, and



$$\rho = \{x \mid a < x < b\}^2 \cup \Delta$$

is a congruence on L.

Now,

$$\rho \wedge L_1^2 = d_{L_1^2}, \text{ and } (\rho \wedge L_1^2)_\Delta = \Delta$$

On the other hand, $\rho_\Delta = \rho$, $L_1^2 = L^2$, and thus $\rho_\Delta \wedge L_{1_\Delta}^2 = \rho$.

Hence, $(\rho \wedge L_1^2)_\Delta \leq \rho_\Delta \wedge L_{1_\Delta}^2$, and the CIP is not fulfilled. □

Using the congruence $\{0,1\}^2$ instead of L_1^2, one can prove that a bounded lattice satisfies the CIP if and only if it is a two-element.

Whether an arbitrary lattice fails on CIP is still an *open problem*.

Theorem 1.5. *An arbitrary lattice L has a modular lattice of weak congruences if and only if it is a two-element chain.*

Proof. Straightforward, by (II), and by Proposition 1.1, 1.2, 1.3 and 1.4. □

2. In this part we shall characterize the lattices having a complemented lattice of weak congruences.

It was proved in [6] that no algebra having less than two different constants can have a complemented lattice of weak congruences. Therefore, we shall look for our lattices in the class of bounded ones, denoted by $(L, \wedge, \vee, 0, 1)$, $0 \neq 1$.

Theorem 2.1. *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. Then it has a complemented lattice of weak congruences if and only if its lattice of sublattices is complemented.*

Proof. Let L be a bounded lattice with a complemented lattice of weak congruences $(C\ (L),\le)$. Then, for $\rho,\rho' \in C\ (L)$,

$$\rho \vee \rho' = L^2, \quad \rho \wedge \rho' = d_{\blacksquare} \quad (\text{where } d_{\blacksquare} = \{0,1\}^2 \wedge \Delta),$$

we have that $d_\rho \vee d_{\rho'} = \Delta$, and $d_\rho \wedge d_{\rho'} = d_{\blacksquare}$. Since the diagonal relation represents a corresponding subalgebra ((*I*)), it follows that $(S(L),\le)$ is a complemented lattice.

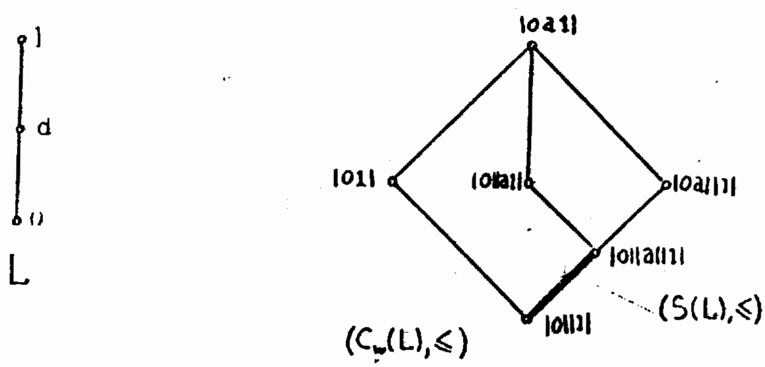
Now let $(S(L),\le)$ be a complemented lattice. Consider $\rho \in C(M)$, $M \in S(L)$, such that $\rho \neq M^2$, and let $N \in S(L)$ be a complement of M in $S(L)$. Then,

$$\rho \vee N^2 = L^2, \quad \rho \wedge N^2 = d_{\blacksquare} \ .$$

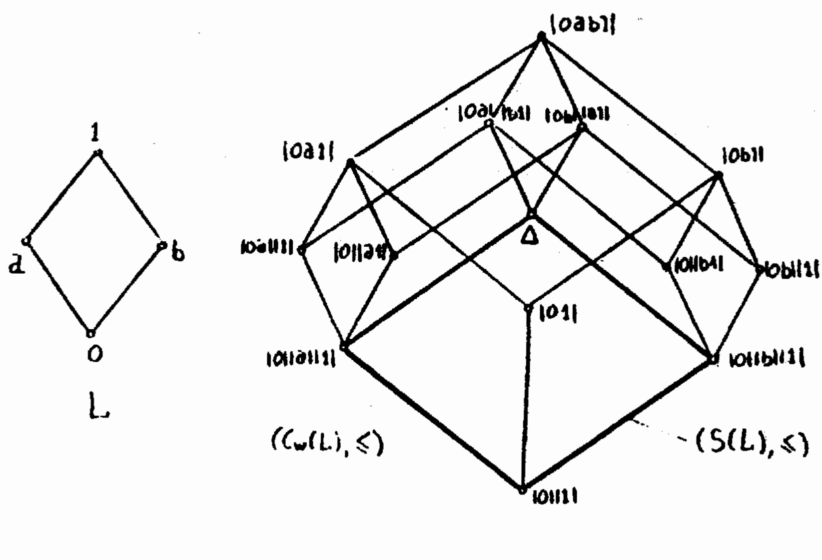
If $\rho = M^2$, its complement is $\rho' \in C(N)$, such that $\rho' \neq N^2$. □

Examples

(i)



(ii)



The lattices in (*i*) and (*ii*) are bounded ($k = \{0,1\}$), and they both have complemented lattices of weak congruences, as well as lattices of sublattices.

Note that the lattice in (*ii*), considered as a Boolean algebra $(B,\wedge,\vee,-,0,1)$, has the same (up to the isomorphisms) lattice of weak congruences as L in (*i*).

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Rezime

Slabe kongruencije mreze

a) da je mreža slabih kongruencija proizvoljne mreže modularna, ako i samo ako je mreža dvoelementna i

b) mreža slabih kongruencija ograničene mreže je komplementirana ako i samo ako je mreža njenih podmreža komplementirana.

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