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ON THE S-ASYMPTOTIC OF TEMPERED AND K_1'-DISTRIBUTIONS PART IV. S-ASYMPTOTIC AND THE ORDINARY ASYMPTOTIC

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ABSTRACT

It is proved that the S-asymptotic of an $f \in L^1_{loc}$ implies its ordinar asymptotic behaviour at infinity under some conditions of monotonicity.

0. For the notation and the basic properties of the S-asymptotic behaviour of an $f \in \mathcal{D}'$ we refer to [3],[4]. We shall repeat here only the definition.

Let $f \in \mathcal{D}'$ and $c(h)$, $h>h_0$, be a continuous positive function. If for some $g \in \mathcal{D}'$, $g \neq 0$,

$$(1) \qquad \lim_{h\to\infty} \left\langle \frac{f(x+h)}{c(h)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle, \quad \forall\phi \in \mathcal{D}$$

then we say that f has the S-asymptotic at infinity with respect to $c(h)$ with the limit g. In this case we write $f(x+h) \overset{s}{\sim} g(x)c(h)$ in $\mathcal{D}'$ at infinity. It follows from (1) that for some $A\neq 0$, $\alpha\neq 0$, and some slowly varying function L

$$(2) \qquad g(x) = Ae^{\alpha x}, \quad c(h) = e^{\alpha x}L(e^h).$$

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Recall, L is slowly varying at infinity if L is measurable and $L(ax)/L(x)\to 1$, $x\to\infty$ for every $a>0$; for the properties of such functions we refer to [2].

If we assume in (1) that $f\in S'$ $(f\in K_1')$ and that this limit exists in $S'(K_1')$, i.e. $\forall\phi\in S$ $(\forall\phi\in K_1)$ then we write $f(x+h)\overset{s}{\sim} g(x)c(h)$ in $S'(K_1')$ at infinity. If (1) exists in S' then $g(x)=A$ because the existence of (1) implies that $g\in S'$, i.e. $\alpha=0$ in (2).

1. PROPOSITION 1. *Let $f\in D'$ and $f(x+h)\overset{s}{\sim}1\cdot c(h)$ in D' at infinity with $c(h)=e^{\alpha h}h^{\beta}L(h)$, $h_0<h$ and monotonous L. Then $\alpha=0$ and any distribution g with $\operatorname{supp} g\subset[a,\infty)$ for some $a\in R$, which is equal to f in a neighbourhood of infinity is from S' and $g(x+h)\overset{s}{\sim}1\cdot c(h)$ in S' at infinity.*

PROOF. That $\alpha=0$ is a direct consequence of (2). Let $\psi\in C^{\infty}$, $\psi=1$ for $x\geq 1$ and $\psi=0$ for $x\leq 0$. We have that $\psi f\overset{s}{\sim}1\cdot c(h)$ in D' and so, that $\left\{\frac{(\psi f)(x+h)}{c(h)},\ h\in R\right\}$ is bounded in D'. [5] implies that $\psi f\in S'$. The assumption on L implies that we can apply [4, Part II, Prop. 2] and [4, Part I] which gives

$$(\psi f)\overset{s}{\sim}1\cdot c(h) \text{ in } S' \text{ at infinity.}$$

Let g satisfy assumptions of the proposition. Clearly, $(\psi f-g)\overset{s}{\sim}0\cdot c(h)$ in D' at infinity and the application of [4, Parts I, II] gives that the same holds in S'.

By using [4, Parts I, II] in the same way as in Proposition 1 we have:

PROPOSITION 2. *Let $f\in D'$, $f\overset{s}{\sim} e^{\alpha x}c(h)$ in D' at infinity, where $c(h)=e^{\alpha h}h^{\beta}L(h)$, $h_0>h$, and L is monotonous. Then $\tilde{\alpha}=\alpha$ and every g with the support bounded from the left and equal to f in some neighbourhood of infinity, is from K_1' and $g(x+h)\overset{s}{\sim} e^{\alpha x}c(h)$ in K_1' at infinity.*

Now we can easily prove:

PROPOSITION 3. *Let* $g \in \mathcal{D}'$, $\operatorname{supp} g \subset [a,\infty)$ *and* L *be monotonuous. The following conditions are equivalent:*

(a) $g(x+h) \overset{s}{\sim} e^{\alpha x} e^{\alpha h} h^{\beta} L(h)$, *in* K_1' *at infinity,*

(b) $e^{-\alpha(x+h)} g(x+h) \overset{s}{\sim} 1 \cdot h^{\beta} L(h)$ *in* S' *at infinity.*

2. The ordinar asymptotic behaviour of an $f \in L^1_{loc}$ at infinity with respect to $c(x)=e^{\alpha x} L(e^x)$ implies, under some simple conditions, the S-asymptotic behaviour of f at infinity in $\mathcal{D}'$ with respect to $e^{\alpha h} L(e^h)$. This is quoted in [3]. The question is: when the S-asymptotic of an $f \in L^1_{loc}$ implies its ordinary asymptotic? First we shall give an example.

On can easily construct a function G such that $G(n)=n^n$, $n \in N$, $G(x)=0$ for $x \notin I_n$, where I_n is a suitable small interval around n and that

$$G_1(x) = \int_0^x G(t)\,dt \to 1 \quad \text{as} \quad x \to \infty.$$

Clearly, $G_1(x+h) \overset{s}{\sim} 1 \cdot 1 \, (g(x)=1 \;\; c(h)=1)$ in $\mathcal{D}'$ at infinity. This implies that $\lim_{h\to\infty} \langle G(x+h), \phi(x)\rangle = 0 \;\; \forall \phi \in \mathcal{D}'$.

Let $f(x)=1+G(x)$, $x \in R$. We have $f(x+h) \overset{s}{\sim} 1 \cdot 1$ in $\mathcal{D}'$ at infinity but $f(x)$ has not the ordinary asymptotic at infinity.

The following proposition gives the sufficient condition under which the S-asymptotic of an $f \in L^1_{loc}$ implies its ordinar asymptotic behaviour.

PROPOSITION 4. *Let* $f \in L^1_{loc}$, $c(h)=h^{\beta} L(h)$, $h>h_o$, $\beta \in R$ *and* L *be monotonuous. If for some* $m_o \in N_o$ *and* $x_o \in R$, $f(x) x^{m_o}$ *is non-decreasing for* $x>x_o$ *then* $\lim_{h\to\infty} \frac{f(h)}{c(h)} = 1$.

PROOF. Let $m \in R$ so that $m \geq m_o$ and $m>-\beta$. Let ψ be as in the proof of Proposition 1. By [2] and Proposition 1 we have

$$((1+t^2)^{m/2} \psi(t) f(t))(x+h) \overset{s}{\sim} 1 \cdot h^{m+\beta} L(h) \;\text{ in } S' \text{ at infinity.}$$

Since m+β>0, we have $(1+t^2)^{m/2}\psi(t)f(t)$ has the quasiasymptotic at infinity with respect to $k^{m+\beta}L(k)$. (For the basic properties of the quasiasymptotic behaviour we refer to [6]). Now from the fact that $(1+t^2)^{m/2}\psi(t)f(t)$ is non-decreasing for $t>x_o$ and that m+β>0, by using [2,Th.2] we get that

$$\lim_{t\to\infty} \frac{(1+t^2)^{m/2}\psi(t)f(t)}{t^{m+\beta}L(t)} = 1.$$

This implies the assertion.

Propositions 3. and 4. imply:

PROPOSITION 5. *Let* $f\in L^1$, $f(x+h)\overset{s}{\sim} e^{\alpha x}e^{\alpha h}h^{\beta}L(h)$, *where* L *is monotonuous. Assume that for some* m_o *and* x_o, $f(x)e^{-\alpha x}x^{m_o}$, $x>x_o$, *is non-decreasing. Then*

$$f(x)\sim e^{\alpha x}x^{\beta}L(x),\quad x\to\infty.$$

REFERENCES

|1| Дрожжинов, .Н., Завьялов,Б.И., Квазиасимптотика обобщенных функций и Тауберовы теоремы в комплексной области, Мат.Сб. 102(144)(1977), 372-390.

|2| Seneta,E., *Regularly varying functions, Lecture Notes in Math., Springer-Verlag, Berlin-Heidelberg-New York, 1976.*

|3| Pilipović,S., Stanković,B., *S-asymptotic of a distribution Pliska (to appear).*

|4| Pilipović,S., *S-asymptotic of tempered and K'-distributions, Part I,II, Rev.Res.Sci.Univ.Novi Sad, 15, N°1 (1985), 47-58, 59-67.*

|5| Schwartz,L., *Theorie des distributions I-II, Hermann, Paris, 1950-1951.951.*

|6| Владимиров,В.С., Дрожжинов, .Н., Завьялов,Б.Ч., Многомерные Тауберовы теоремы для обобщенных функций, Наука, Москва, 1986.

REZIME

O S-ASIMPTOTICI TEMPERIRANIH I K_1'-DISTRIBUCIJA.
IV DEO. S-ASIMPTOTIKA I OBIČNA ASIMPTOTIKA.

Dati su uslovi na lokalno-integrabilnu funkciju koja ima S-asimptotsko ponašanje u ∞ koji impliciraju njeno obično asimptotsko ponašanje.

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