Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 18,2, 179–189 (1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

ON THE S-ASYMPTOTIC OF TEMPERED AND K_1'-DISTRIBUTIONS. PART III, STRUCTURAL THEOREMS

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ABSTRACT

Several structural properties of a distribution f which has the S-asymptotic behaviour are given.

1. It is said ([5]) that an $f \in \mathcal{D}'(\mathbb{R})$ has the S-asymptotic at ∞ with respect to a continuous and positive function $c(h)$, $h \in (A,\infty)$, $A>0$, if for some $g \in \mathcal{D}'(\mathbb{R})$

$$(1) \qquad \lim_{h\to\infty} \left\langle \frac{f(x+h)}{c(h)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle, \quad \forall \phi \in \mathcal{D}.$$

In this case we write $f(x+h) \overset{s}{\sim} g(x)c(h)$, $h\to\infty$.

It is shown in [5] that g must be of the form

$$(2) \qquad g(x) = C\exp(\alpha x), \; C \in \mathbb{R}, \; \alpha \in \mathbb{R},$$

and if in (2) $C \neq 0$ then c must be of the form

$$c(h) = \exp(\alpha h)L(\exp h), \; h > A,$$

where L is a slowly varying function ([7]). With no loss of

AMS Mathematics Subject Classification (1980): 46F05
Key words and phrases: Structural theorems, tempered distributions, asymptotic behaviour.

generality, we shall always assume that L is continuous and different from zero in $(1,\infty)$. For the properties of the S-asymptotic behaviour of distributions we refer the reader to [2],[3],[4],[5] and [8].

We shall give in this note several structural properties of an f which has the S-asymptotic.

2. THEOREM 1. *Let* $f \in L^1_{loc}$,

$$f(x) \sim \exp(\alpha x) L(\exp x), \quad x \to \infty \textit{ (in the ordinary sense).}$$

Then

$$f(x+h) \overset{s}{\sim} \exp(\alpha x)\exp(\alpha h) L(\exp h) \textit{ as } h \to \infty .$$

PROOF. Since $L(\lambda h)/L(h)$ converges uniformly to 1 $(h \to \infty)$ on any interval $[a,b] \subset (0,\infty)$ ([7]) we have ($\phi \in \mathcal{D}$, supp$\phi \subset [a,b]$)

$$\int_{-\infty}^{\infty} \frac{f(x+h)\phi(x)\,dx}{\exp(\alpha h) L(\exp h)}$$

$$= \int_a^b \frac{f(x+h)\exp(\alpha x)\phi(x)}{\exp(\alpha(x+h)) L(\exp(x+h))} \, \frac{L(\exp x \exp h)}{L(\exp h)}\,dx$$

$$\to \int_a^b \exp(\alpha x)\phi(x)\,dx, \quad h \to \infty.$$

This proves the assertion.

THEOREM 2. *Let* $f(x+h) \overset{s}{\sim} 1 \cdot h^{\nu} L(h)$, $h \to \infty$, *where* $\nu > -1$. *Let* $g \in \mathcal{D}'$ *and for some* $m \in \mathbb{N}$, $g^{(m)} = f$. *Then*

$$g(x+h) \overset{s}{\sim} 1 \cdot h^{\nu+m} L(h), \quad h \to \infty . \tag{3}$$

PROOF. By the l'Hospital rule, we obtain

$$\lim_{h\to\infty} \frac{\langle g^{(m-1)}(x+h), \phi(x)\rangle}{\int_1^h t^{\nu} L(t)\,dt} \to \langle 1,\phi\rangle, \quad \forall \phi \in \mathcal{D}.$$

From [7,Ch.2], it follows that

$$\int_1^h t^{\nu} L(t)\,dt \sim h^{\nu+1} L(h),\ h\to\infty.$$

So, the proof of the theorem follows by repeating the l'Hospital rule m-times.

Note that we can formulate and prove a similar assertion as in Theorem 2, for $\nu\leq -1$, if we assume the supplement conditions which enable us the use of the l'Hospital rule.

The generalized form of Theorem 2 is the following one:

THEOREM 2'. *Let* $f(x+h) \overset{s}{\sim} (\exp \alpha x)\exp(\alpha h)L(\exp h)$, $h\to\infty$, *and*

$$\int_0^x \exp(\alpha h)L(\exp h)\,dh\to\infty,$$ *as* $x\to\infty$. *Let* $g\in\mathcal{D}'$ *such that*

$g^{(m)}=f$ *for some* $m\in\mathbb{N}$. *Then*

$$g(x+h) \overset{s}{\sim} (\exp\alpha x)\int_0^h (\ldots((\int_0^{h_1}\exp(\alpha t)L(\exp t)dt)dh_1)\ldots)\cdot dh_{m-1},\ h\to\infty.$$

If we assume on L instead of continuity that is measurable the following theorem is of interest:

THEOREM 3. *Let* $\phi_0\in C_0^{\infty}$ *such that* $\int\phi_0(t)dt=1$. *If* *and*

$$\lim_{h\to\infty} \left\langle \frac{f^{(i)}(x+h)}{\exp(\alpha h)L(h)},\ \phi_0(x)\right\rangle = (\alpha)^i\langle\exp(\alpha x),\ \phi_0(x)\rangle\ ,$$

$$i=0,1,\ldots,m-1$$

and

$$f^{(m)}(x+h) \overset{s}{\sim} \alpha^m \exp(\alpha x)(\exp(\alpha h))L(\exp h),\ h\to\infty,$$

then

$$f(x+h) \overset{s}{\sim} \exp(\alpha x)(\exp(\alpha h))L(\exp h),\quad h\to\infty.$$

PROOF. This asseartion was proved in [2] for m=1. By repeating the same arguments, the proof for m>1 follows. Here is the proof for m=1.

Any $\phi \in \mathcal{D}$ can be written in the form

$\phi(x)=\phi_0(x)\int_{-\infty}^{\infty}\phi(x)dx+\tilde{\phi}(x)$, where $\tilde{\phi}\in\mathcal{D}$ such that $\int_{-\infty}^{\infty}\tilde{\phi}(t)dt=0$.

Obviously,

$$\tilde{\phi}(x)=\Big(\int_{-\infty}^{x}\tilde{\phi}(t)dt\Big)' \text{ where } x\to\int_{-\infty}^{x}\tilde{\phi}(t)dt\in\mathcal{D},\ x\in\mathbb{R}.$$

So, for any $\phi\in\mathcal{D}$ we have

$$\lim_{h\to\infty}\Big\langle\frac{f(x+h)}{\exp(\alpha h)L(\exp h)},\ \phi(x)\Big\rangle =$$

$$=\lim_{h\to\infty}\Big\langle\frac{f(x+h)}{\exp(\alpha h)L(\exp h)},\ \phi_0(x)\Big\rangle\int_{-\infty}^{\infty}\phi(x)dx\ +$$

$$+\lim_{h\to\infty}\Big\langle\frac{f(x+h)}{\exp(\alpha h)L(\exp h)},\ \Big(\int_{-\infty}^{x}\tilde{\phi}(t)dt\Big)'\Big\rangle .$$

Now, assumptions of the theorem imply the assertion.

3. For the main structural theorem we need the following lemma which was proved in [2]. For the sake of completeness, we shall give here the complete proof of it.

LEMMA 4. *Let* $c(h)$, $h\in(0,\infty)$, *be a real-valued positive locally integrable function such that for some* $f\in\mathcal{D}'$ *the limit in (1) exists with* $g\neq 0$. *There exist* $c\in C^{\infty}$ *different from zero on* $\mathbb{R}$, $\alpha\in\mathbb{R}$ *and* $A\in\mathbb{R}$, $A\neq 0$, *such that*

$$c(h)/\tilde{c}(x+h)\to A^{-1}\exp(-\alpha x),\ h\to\infty,$$

in the sense of convergence in E.

PROOF. Let $c_0(x)=c(x)$ for $x>1$, $c_0(x)=1$ for $x\leq 1$, and $\omega\in C_0^{\infty}$ such that

$$\operatorname{supp}\omega\subset[-1,1],\ \omega(x)>0 \text{ for } x\in(-1,1), \text{ and } \int_{-1}^{1}\omega(t)dt=1.$$

We put $\tilde{c}(x)=(c_o(t)*\omega(t))(x)$, $x\in\mathbb{R}$. Obviously $\tilde{c}\in C^\infty$. Since for some $\varepsilon\in(0,1)$,

$$\int_{-1}^{1} c_o(x-t)\omega(t)dt \geq \int_{-\varepsilon}^{\varepsilon} c_o(x-t)\omega(t)dt \geq \min\{\omega(t);\ |t|\leq\varepsilon\}\cdot$$

$$\cdot\int_{-\varepsilon}^{\varepsilon} c_o(x-t)dt>0,$$

we obtain that $\tilde{c}(x)\neq 0$, $x\in\mathbb{R}$.

Let K be a compact set in $\mathbb{R}$. There exists $\alpha\in\mathbb{R}$ such that for any $x\in K$ and $t\in[-1,1]$

$$c_o(x+h-t)/c(h)\to\exp(\alpha(x-t)),\ h\to\infty,\quad ([5,\ \text{Theorem 3a})]).$$

Since the set $K_o=\{x-t,\ x\in K,\ t\in[-1,1]\}$ is a compact one, the last convergence is uniform on K_o. Let us prove it.

Because in (1) $g\neq 0$, we obtain that for some $\phi\in C_o^\infty$, $m=\langle g,\phi\rangle\neq 0$. Let $y\in K_o$, $h>\max\{1,(1-\min\{t;t\in K_o\})\}$. We put

$$d_h(y)=c_o(h+y)/c(h);\quad r_h(y)=\langle f(x+h+y)/c_o(h+y),\ \phi(x)\rangle;$$

$$s_h(y) = \langle f(x+h+y)/c(h),\ \phi(x)\rangle.$$

We have

$$d_h(y)\to\exp(\alpha y),\ h\to\infty;$$

$$r_h(x)\to m\neq 0,\ h\to\infty,\ \text{uniformly on } K_o;$$

$$r_h(y)d_h(y)=s_h(y)\to\langle g(x+y),\phi(x)\rangle = m\exp(\alpha y),\ h\to\infty,$$

uniformly on K_o because the set $(x-y);\ y\ \ K_o$ is bounded in *D* and the strong and weak convergence are equivalent in *D'*.

Using the inequality

$$|d_h(y)r_h(y)-m\exp(\alpha y)|\geq |r_h(y)||d_h(y)-\exp(\alpha y)|-$$

$$-\exp(\alpha y)|r_h(y)-m|,$$

one can easily prove that if $d_h(y)$ does not converge uniformly to $\exp(\alpha y)$ on K_o, then $s_h(y)$ does not converge uniformly to $m\exp(\alpha y)$ on K_o as $h\to\infty$. This is a contradiction. Thus we have proved that

$$c_o(x+h-t)/c(h) \to \exp(\alpha(x-t)), \quad h\to\infty,$$

uniformly in $x\in K$, $t\in[-1,1]$. This implies that for a non-negative integer β

$$\tilde{c}^{(\beta)}(x+h)/c(h) = \int_{-1}^{1}(c_o(x+h-t)/c(h))\omega^{(\beta)}(t)dt$$

$$\to \int_{-1}^{1}\exp\alpha(x-t)\omega^{(\beta)}(t)dt, \quad h\to\infty,$$

uniformly on K, i.e.

$$\tilde{c}^{(\beta)}(x+h)/c(h) \to A(\alpha)^{\beta}\exp(\alpha x),$$

where $A = \int_{-1}^{1}\exp(-\alpha t)\omega(t)dt$, $h\to\infty$,

uniformly in $x\in K$. By the same arguments, one can prove that

$$c(h)/\tilde{c}(x+h) \to A^{-1}\exp(-\alpha x), \quad h\to\infty,$$

uniformly on K.
Now, by induction, one can prove that for every non-negative integer

$$(c(h)/\tilde{c}(x+h))^{(\beta)} \to (A^{-1}\exp(-\alpha x))^{(\beta)}, \quad h\to\infty,$$

uniformly on K.

THEOREM 5. *Let* $f(x+h)\overset{s}{\sim}(\exp\alpha x)(\exp\alpha h)L(\exp h)$, $h\to\infty$, *where* α *is different from* 0. *There is* $m_o\in\mathbb{N}$ *such that for every* $m\geq m_o$ *there are* $g_{m,i}\in C(1,\infty)$, $i=0,1,\ldots,m$ *such that*

$$f(x) = \sum_{i=o}^{m} g_{m,i}^{(i)}(x), \quad x\in(1,\infty),$$

and

$$g_{m,i}(x)\sim C_i x^m\exp(\alpha x)L(\exp x), \quad x\to\infty,$$

where C_i *are suitable constants.*

($C(1,\infty)$ is the space of all the continuous functions on $(1,\infty)$.)

PROOF. The function $c(h)=\exp(\alpha h)L(\exp h)$, $h>A$, satisfies the conditions of Lemma 4. Let $\tilde{c}\in C^{\infty}$ correspond to this function.

From [6, T.I., p.72, Théoréme X] we obtain

$$\lim_{h\to\infty} < \frac{f(x+h)}{\tilde{c}(x+h)}, \phi(x) > = \lim_{h\to\infty} < \frac{f(x+h)}{c(h)}, \frac{c(h)}{\tilde{c}(x+h)} \phi(x) > =$$

$$= < 1,\phi(x)>, \ \forall\phi\in\mathcal{D}.$$

Let $\theta\in C^{\infty}$, $\theta(x)=0$ for $x<0$ and $\theta(x)=1$ for $x>1$. We have

$$\frac{\theta(\cdot+h)f(\cdot+h)}{\tilde{c}(\cdot+h)} \to 1 \text{ in } \mathcal{D}' \text{ as } h\to\infty.$$

Thus, $\{\frac{\theta(\cdot+h)f(\cdot+h)}{\tilde{c}(\cdot+h)}, h>0\}$ is a bounded subset of $\mathcal{D}'$. This implies that this set is bounded in S'. By the Banach-Steinhaus Theorem we obtain

$$\frac{\theta(\cdot+h)f(\cdot+h)}{\tilde{c}(\cdot+h)} \to 1 \text{ in } S' \text{ as } h\to\infty, \text{ i.e.}$$

$$\lim_{h\to\infty} < \frac{(\theta f/\tilde{c})(x+h)}{d(h)}, \phi(x)> = <1,\phi>, \ \forall\phi\in S,$$

where $d(h)=1, h>A$. Since the S-asymptotic in S'_+ with $\nu>-1$ (in our case $d(h)=h^{\nu}, h>A$, $\nu=0$) implies the quasiasymptotic of $\theta f/\tilde{c}$, the structural theorem [1, Theorem I] implies that there is m_o such that for every $m>m_o$ there is $F_m\in C(-\infty,\infty)$ such that

$$(\theta f/\tilde{c})(x) = F^{(m)}(x), \ x\in\mathbb{R}, \text{ and}$$

$$F_m(x) \sim x^m \text{ as } x\to\infty.$$

Thus, we obtain

$$f(x) = \tilde{c}(x)F_m^{(m)}(x), \ x\in(1,\infty).$$

The Leibniz formula implies

$$f(x) = \sum_{i=o}^{m} \binom{m}{i}(-1)^i(\tilde{c}^{(i)}(x)F_m(x))^{(m-i)}, \ x\in(1,\infty).$$

Since

$$\frac{\tilde{c}^{(i)}(x+h)}{c(h)} \to A(\alpha)^{i} e^{x}, \quad h \to \infty \quad (x \in \mathbb{R}),$$

we obtain

$$\tilde{c}^{(i)}(h) \sim A(\alpha)^{(i)} c(h), \quad h \to \infty.$$

This implies the proof.

Now, observe the case $\alpha=0$.

THEOREM 6. *Let* $f(x+h) \overset{s}{\sim} 1 \cdot h^{\nu} L(h)$ *with* $\nu>-1$. *Then there is* $m_o \in \mathbb{N}$ *such that for every* $m>m_o$ *there is* $F_m \in C(1,\infty)$ *such that*

$$f = F_m^{(m)}$$

and

$$F_m(x) \sim x^{m+\nu} L(x), \quad x \to \infty.$$

PROOF. For $\nu>-1$ the S-asymptotic of θf (θ is defined in the preceding proof) implies the quasiasymptotic of this distribution with respect to $h^{\nu}L(h)$. Now, [1, Theorem T] implies the assertion.

THEOREM 7. *Let* $f(x+h) \sim 1\ h^{\nu} L(h)$, *where* $\nu<-1$. *Then there is* $m_o \in \mathbb{N}_o$ *such that for every* $m>m_o$ *there are* $f_{m,i} \in C(1,\infty)$ *and* $A_{m,i} \neq 0$, $i=0,\ldots,m$, *such that*

$$f_{m,i}(x) \sim A_{m,i} x^{m+\nu-i} L(x), \quad i=0,1,\ldots,m$$

and

$$f(x) = \sum_{i=o}^{m} f_{m,i}^{(m-i)}(x), \quad x \in (1,\infty).$$

PROOF. Take $k>0$ such that $k+\nu>-1$. With θ as in the preceding proof, we have

$$(1+(x+h)^2)^{k/2} \theta(x+h) f(x+h) \overset{s}{\sim} 1\ h^{k+\nu} L(h), \quad h \to \infty.$$

By the same arguments as in the preceding proof, we have that there is m_o such that for every $m>m_o$ there is an $F \in C(-\infty,\infty)$, $\operatorname{supp} F \subset [0,\infty)$

$$F_m(x) \sim x^{\nu+k+m}L(x), \quad x\to\infty$$

and

$$(1+x^2)^{k/2}\theta(x)f(x) = F_m^{(m)}(x), \quad x\in\mathbb{R}.$$

Thus, for $x\in(1,\infty)$ we have

$$f(x) = \sum_{i=0}^{m}\binom{m}{i}(-1)^i\left(\left(\frac{1}{(1+x^2)^{k/2}}\right)^{(i)}F_m(x)\right)^{(m-i)}$$

The proof follows from the fact that

$$\left(\frac{1}{(1+x^2)^{k/2}}\right)^{(i)} \sim C_i x^{-k-i}, \quad x\to\infty,$$

where $C_i\neq 0$ are suitable constants, $i=0,\dots,m$.

The more difficult problem is the following one.

Let $f(x+h) \overset{s}{\sim} g(x)c(h)$, $h\to\infty$, $g\neq 0$. There is a question whether $f'(x)$ has the S-asymptotic with the limit $g_1\neq 0$ with respect to some $c_1(h)$. In many special cases, the answer can be given easily, but for example, if $c(h)=h^\nu$, we do not know any satisfactory answer.

The same problem in "classical" analysis is even more difficult. Namely, for an $f\in C^1$, it can happen that f has tha S-asymptotic behaviour with respect to some $c(h)$ but f' does not have the ordinary asymptotic bahaviour with respect to $c(h)$. This is shown by the following example.

Suppose that $F\in L^1$, $F\geq 0$ and that for some sequence (ε_i), $\varepsilon_i>0$ and (x_i), $x_{i+1}>x_i>i$, $F(x)=\exp(\exp x_i)$, $x\in(x_i-\varepsilon_i, x_i+\varepsilon_i)$, $i\in\mathbb{N}$. Let $G(x) = \int_{-\infty}^{x}F(t)dt$. We have

$$(\exp tG(t))(x+h) \overset{s}{\sim} A\exp x\exp h \quad \text{as} \quad h\to\infty,$$

where

$$A = \int_{-\infty}^{\infty}F(t)dt.$$

This implies that

$$(\exp tG(t))'(x+h) \overset{s}{\sim} A\exp x\exp h \quad \text{as} \quad h\to\infty.$$

Obviously, $(\exp x\,(G(x))'$ does not have the ordinary asymptotic behaviour as the function $\exp x$ when $x \to \infty$.

Let us note that the distribution

$$f(x) = x^2 + x \sin x, \; x \in \mathbb{R},$$

satisfies the relation

$$f(x+h) \overset{s}{\sim} 1\, h^2 \quad \text{as} \quad h \to \infty,$$

but its derivative does not have the S-asymptotic behaviour with the limit distribution different from 0.

REFERENCES

[1] Дрожжинов, .Н., Завьялов,Б.И., Квазиасимптотика обошенных функции и тауберовы теоремы в комплексной области, Матем. Сб., 102 (1977), 372-390.

[2] Pilipović,S., *On the Behaviour of Distributions at Infinity, Proc. Conf. Math. Anal., Szyrk., 1985. (to appear).*

[3] Pilipović,S., *Remarks on Supports of Distributions, Gl. Math., 22(42) (1987), 375-380.*

[4] Pilipović,S., *On the S-asymptotics of Tempered and K_1'-Distributions, Parts I and II, Rev. Res. Sci. Math. Univ. Novi Sad, 15, 47-58, 59-67.*

[5] Pilipović,S., Stanković,B., *S-asymptotic of Schwartz Distributions, Pliska (to appear).*

[6] Schwartz,L., *Théorie des Distributions, Hermann, Paris, T.I. (1957), T.II (1957).*

[7] Seneta,E., *Regularly Varying Functions, Lect. Not. Math. 508, Springer, Berlin-Heidelberg-New York, 1976.*

[8] Stanković,B., *Applications of the S-asymptotic, Rev.Res. Sci. Math. Univ. Novi Sad, 15, 1-9.*

REZIME

O S-ASIMPTOTICI TEMPERIRANIH I K_1'-DISTRIBUCIJA.
DEO III, STRUKTURNE TEOREME

Dato je nekoliko strukturnih osobina distribucije f koja ima S-asimptotsko ponašanje.

Received by the editors June 1,1986.