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COMMON FIXED POINT THEOREMS FOR SINGLE-VALUED AND MULTIVALUED MAPPINGS

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Abstract

Using the notion of a compatible pair of mappings some generalizations of common fixed point theorems from [2] and [5] are proved.

1. Introduction

S. Sessa introduced in [6] the notion of a weakly commuting pair of mappings in the following way.

DEFINITION 1 *Let* (X,d) *be a metric space and* $f,g\colon X \to X$. *Then the pair* (f,g) *is said to be weakly commuting if and only if for every* $x \in X$

$$d(fgx, gfx) \le d(fx, gx).$$

Every commuting pair is obviously weakly commuting but the converse is false [6].

G. Jungck generalized the notion of a weakly commuting pair in the following way.

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DEFINITION 2 [3] *Let* (X,d) *be a metric space and* $f,g: X \to X$. *Then the pair* (f,g) *is said to be compatible if and only if for every sequence* $\{x_n\}_{n\in N}$ *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, *for some* $t \in X$:

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

If f and g are such that $\lim_{n\to\infty} d(fx_n,gx_n) = 0 \Rightarrow \lim_{n\to\infty} d(fgx_n,gfx_n) = 0$, then the pair (f,g) is obviously compatible. Hence, every weakly commuting pair is compatible, but the converse is false [3].

We shall generalize the notion of a compatible pair to the case when g is a multivalued mapping. By $CB(X)$ the collection of all closed and bounded subsets of X is denoted.

DEFINITION 3 *Let* (X,d) *be a metric space,* $f: X \to X$ *and* $g: X \to CB(X)$. *We say that the pair* (f,g) *is compatible if and only if for every two sequences* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *from* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = t$ *for some* $t \in X$, *where* $y_n \in gx_n$, $n \in N$:

$$\lim_{n\to\infty} d(fy_n, gfx_n) = 0.$$

In [4] the notion of a weakly commuting pair (f,g), is introduced where $f: X \to X$ and $g: X \to CB(X)$. The pair (f,g) is weakly commuting if and only if for each $x \in X$, $fgx \in CB(X)$ and

$$H(gfx, fgx) \le d(fx, gx),$$

where H is the Hausdorff metric defined on $CB(X)$.

Suppose that the pair (f,g) is weakly commuting, $f: X \to X$ and $g: X \to CB(X)$ and prove that it is compatible.

Let $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ be two sequences from X such that $y_n \in gx_n$, $n \in N$ and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = t \in X.$$

From $d(fx_n,gx_n) \le d(fx_n,y_n)$ it follows that $\lim_{n\to\infty} d(fx_n,gx_n) = 0$. Hence $\lim_{n\to\infty} H(gfx_n,fgx_n) = 0$ and since $d(fy_n,gfx_n) \le H(fgx_n,gfx_n)$ we obtain that $\lim_{n\to\infty} d(fy_n,gfx_n) = 0$ which means that the pair (f,g) is compatible in the sense of Definition 3.

In this paper we shall prove some generalizations of common fixed point theorems from [2], [4], [5].

Common fixed point theorems

Let $h: [0,\infty)^5 \to [0,\infty)$ be a nondecreasing and upper semicontinuous mapping in each variable. We say that h is from the class Ψ if and only if for every $t > 0$:

$$t > \varphi(t) = \max\{h(t,0,0,t,t),\ h(t,t,t,2t,0),\ h(t,t,t,0,2t)\}.$$

THEOREM 1 *Let* (X,d) *be a complete metric space*, $S,T,f,g\colon X \to X$, $TX \subset fX$, $SX \subset gX$ *and for all* $x,y \in X$:

(1) $d(Sx,Ty) \le h(d(fx,gy),\ d(fx,Sx),\ d(gy,Ty),\ d(fx,Ty),\ d(gy,Sx))$

where h is from the class Ψ.

If one of the mappings S,T,f and g is continuous and the pairs (S,f) and (T,g) are compatible, then S,T,f and g have a common fixed point z and z is the unique common fixed point of S and f and T and g.

Proof: As in [2], for an arbitrary $x_0 \in X$ we can define a sequence $\{x_n\}_{n\in N\cup\{0\}}$ such that $Sx_{2n} = gx_{2n+1}$, $Tx_{2n+1} = fx_{2n+2}$, $n \in N \cup \{0\}$ and prove that the sequence $\{Sx_0, Tx_1, Sx_2, Tx_3, \ldots\}$ converges to a point $z \in X$. Then

$$z = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} fx_{2n}$$

Using the similar method as in [5] we shall prove that $z = Sz = Tz = gz = fz$.

a) Suppose that f is continuous.

From (1) we have

(2) $d(Sfx_{2n},Tx_{2n+1}) \le h(d(f^2x_{2n},gx_{2n+1}),d(f^2x_{2n},Sfx_{2n}),\ d(gx_{2n+1},Tx_{2n+1}),$
$d(f^2x_{2n},Tx_{2n+1}),\ d(gx_{2n+1},Sfx_{2n}))$.

Further, since f is continuous we obtain that

$$\lim_{n\to\infty} fSx_{2n} = \lim_{n\to\infty} f^2x_{2n} = fz$$

From the compatibility of the pair (S,f) and the relations

$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Sx_{2n} = z$ we have that $\lim_{n\to\infty} d(Sfx_{2n},fSx_{2n}) = 0$

and so from (2) we obtain that

$$d(fz,z) \le h(d(fz,z),\ d(fz,fz),\ d(z,z),\ d(fz,z)\ d(fz,z))$$
$$= h(d(fz,z),\ 0,0,\ d(fz,z),\ d(fz,z)) \le \varphi(d(fz,z)).$$

This implies that $fz=z$. From the inequality $d(Sz,Tx_{2n+1}) \le h(d(fz,gx_{2n+1}), d(fz,Sz), d(gx_{2n+1},Tx_{2n+1}), d(fz,Tx_{2n+1}), d(gx_{2n+1},Sz))$ it follows that $Sz = z$. Since $SX \subset gX$ it follows that there exists $w \in X$ such that $gw = z$.

From (1) we have

$d(z,Tw) = d(Sz,Tw) \le h(d(fz,gw), d(fz,Sz), d(gw,Tw), d(fz,Tw), d(gw,Sz)) = h(d(z,z), d(z,z), d(z,Tw), d(z,Tw), d(z,z)) \le \varphi(d(z,Tw))$ and so $Tw = z$. Since the pair (T,g) is compatible and $Tw = gw = z$, if we take in definition 2 that $x_n=w$, $n \in N$ then we obtain that

$$\lim_{n\to\infty} d(Tgx_n,gTx_n) = d(Tz, gz) = 0.$$

From $Tz = gz$ it follows that $d(z,Tz) = d(Sz,Tz) \le \varphi(d(z,Tz))$ and so $z = Tz = gz = fz = Sz$.

The proof is similar if g is continuous.

b) Suppose that S is continuous.

The relation $\lim_{n\to\infty} d(Sfx_{2n},fSx_{2n}) = 0$ and the continuity of S implies that $\lim_{n\to\infty} fSx_{2n} = Sz$. From the inequality

$$d(S^2x_{2n},Tx_{2n+1}) \le h(d(fSx_{2n},gx_{2n+1}), d(fSx_{2n},S^2x_{2n}),d(gx_{2n+1}Tx_{2n+1}),$$
$$d(fSx_{2n},Tx_{2n+1}), d(gx_{2n+1},S^2x_{2n}))$$

we obtain that $d(Sz,z) \le \varphi(d(Sz,z))$ and so $Sz=z$.

Let $w' \in X$ so that $gw'=z$. Then from the inequality

$$d(S^2x_{2n},Tw') \le h(d(fSx_{2n},gw'), d(fSx_{2n},S^2x_{2n}), d(gw',Tw'),$$
$$d(fSx_{2n},Tw'), d(gw',S^2x_{2n}))$$

we obtain that $Tw'=z$, which implies, as in a), that $Tz = gz$.

The inequality

$$d(Sx_{2n},Tz) \le h(d(fx_{2n},gz), d(fx_{2n},Sx_{2n}), d(gz,Tz),$$
$$d(fx_{2n},Tz), d(gz,Sx_{2n}))$$

implies that $d(z,Tz) = 0$ and hence $z = Tz = gz = Sz$.

Let $w'' \in X$ so that $fw''= z$. Then from

$$d(Sw'',z) = d(Sw'',Tz) \le \varphi(d(z,Sw''))$$

we obtain that $Sw'' = fw'' = z$. Using the compatibility of the pair (S,f), as in a), it follows that

$$d(Sz,fz) = d(Sfw'',fSw'') = 0$$

and so $Sz = fz = z$.

If T is continuous the proof is similar. It is easy to prove the z is the unique common fixed point of S and f and T and g.

THEOREM 2 *Let (X,d) be a complete metric space, $S,T\colon X \to CB(X)$ H-continuous mappings, $f,g\colon X \to X$ continuous mappings such that $TX \subset fX$, $SX \subset gX$ and*

$$(3) \qquad H(Sx,Ty) \le r\max\{d(fx,gy),\ d(fx,Sx),\ d(gy,Ty),\ \tfrac{1}{2}[d(fx,Ty) + d(gy,Sx)]\},\ \text{for every } x,y \in X$$

where $r \in (0,1)$. If the pairs (f,S) and (g,T) are compatible then there exists $z \in X$ such that $fz \in Sz$ and $gz \in Tz$.

Proof: We shall start with a usual construction.

Let x_0 be an arbitrary element from X. Since $Sx_0 \subset gX$ there exists $x_1 \in X$ such that $gx_1 \in Sx_0$. From $0 < r < 1$ it follows that $\frac{1}{\sqrt{r}} > 1$ and so there exists $y \in Tx_1$ such that

$$d(gx_1,y) \le \frac{1}{\sqrt{r}} H(Sx_0,Tx_1).$$

Since $Tx_1 \subset fX$ there exists $x_2 \in X$ such that $y = fx_2$ and so we have that

$$d(gx_1,fx_2) \le \frac{1}{\sqrt{r}} H(Sx_0,Tx_1).$$

Similarly, there exists $x_3 \in X$ such that $gx_3 \in Sx_2$ and

$$d(gx_3,fx_2) \le \frac{1}{\sqrt{r}} H(Sx_2,Tx_1).$$

Continuing in this way we obtain a sequence $\{x_n\}_{n\in N}$ such that $fx_{2n} \in Tx_{2n-1}$, $gx_{2n+1} \in Sx_{2n}$ and

$$d(gx_{2n+1},fx_{2n}) \le \frac{1}{\sqrt{r}} H(Sx_{2n},Tx_{2n-1}),\ n \in N$$

$$d(gx_{2n+1},fx_{2n+2}) \le \frac{1}{\sqrt{r}} H(Sx_{2n},Tx_{2n+1}),\ n \in N \cup \{0\}.$$

We shall prove, in a standard way, that the sequence $\{gx_1,fx_2,gx_3,fx_4,\ldots\}$ is a Cauchy sequence.

From (3) we have

$$d(gx_{2n+1},fx_{2n+2}) \le \sqrt{r}\max\{d(fx_{2n},gx_{2n+1}),\ d(fx_{2n},Sx_{2n}),$$

$$d(gx_{2n+1},Tx_{2n+1}),\ \frac{1}{2}[d(fx_{2n},Tx_{2n+1}) + d(gx_{2n+1},Sx_{2n})]\}$$

and since $gx_{2n+1}\in Sx_{2n}$ and $fx_{2n+2}\in Tx_{2n+1}$ we obtain that

$$d(gx_{2n+1},fx_{2n+2}) \le \sqrt{r}\max\{d(fx_{2n},gx_{2n+1}),\ d(fx_{2n},gx_{2n+1}),$$

$$d(gx_{2n+1},fx_{2n+2}),\ \frac{1}{2}[d(fx_{2n},fx_{2n+2})]\} \le \sqrt{r}\max\{d(fx_{2n},gx_{2n+1}),$$

$$d(fx_{2n},gx_{2n+1})\ d(gx_{2n+1},fx_{2n+2}),\ \frac{1}{2}[d(fx_{2n},gx_{2n+1}) + d(gx_{2n+1},fx_{2n+2})]\}$$

$$\le \sqrt{r}\max\{d(fx_{2n},gx_{2n+1}),\ d(gx_{2n+1},fx_{2n+2})\}.$$

As in [4] we obtain that

$$d(gx_{2n+1},fx_{2n+2}) \le \sqrt{r}\, d(fx_{2n},gx_{2n+1})$$

and similarly that

$$d(gx_{2n+3},fx_{2n+2}) \le \sqrt{r}\, d(fx_{2n+2},gx_{2n+1}),\quad n\in N.$$

Hence $\{gx_1, fx_2, gx_3, fx_4,\dots\}$ is a Cauchy sequence and let

$$z = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n}.$$

We shall prove that $fz\in Sz$ i.e. that $d(fz,Sz) = 0$. For every $n\in N$ we have

$$d(fgx_{2n+1},Sz) \le d(fgx_{2n+1},Sfx_{2n}) + H(Sfx_{2n},Sz)$$

and we shall prove that $\lim_{n\to\infty} d(fgx_{2n+1},Sz) = 0$.

For this purpose we shall show that $\lim_{n\to\infty} d(fgx_{2n+1},Sfx_{2n}) = 0$. The relation $\lim_{n\to\infty} H(Sfx_{2n},Sz) = 0$ follows from the H-continuity of S. The pair (f,S) is compatible and since for $x'_n = x_{2n}$ and $y_n = gx_{2n+1}$ we have that $\lim_{n\to\infty} fx'_n = \lim_{n\to\infty} y_n = z$ and $y_n\in Sx_{2n}$, it follows that

$$\lim_{n\to\infty} d(fy_n,Sfx'_n) = \lim_{n\to\infty} d(fgx_{2n+1},Sfx_{2n}) = 0.$$

Hence $\lim_{n\to\infty} d(fgx_{2n+1},Sz) = 0$ and so from

$$d(fz,Sz) \le d(fz,fgx_{2n+1}) + d(fgx_{2n+1},Sz)$$

and the continuity of f we obtain that $d(fz,Sz) = 0$ which implies that $fz \in Sz$. Similarly, we can prove that $gz \in Tz$.

References

1. Shih-sen Chang, A common fixed point theorem for commuting mapping, Math. Jap., 26 (1981), 121-129.

2. Xieping Ding, Some common fixed point theorems for commuting mappings, Math. Seminar Notes 11 (1983), 301-305.

3. Gerald Jungck, Compatible mappings and common fixed point, Internat. J. Math. Math Sci. 9(4) (1986), 771-779.

4. Hideaki Kaneko, A common fixed point of weakly commuting multi-valued mappings, Math. Japonica, 33, No. 5 (1988), 741-744.

5. V. Popa, A common fixed point theorem of weakly commuting mappings, Publ. Inst. Math. (in print).

6. S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math., Beograd, 32 (46) (1982), 146-153.

7. C.C. Yeh, On common fixed point theorems of continuous mappings, Indian J. Pure Appl. Math., 10 (1979), 415-420.

Rezime

TEOREME O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI
ZA JEDNOZNAČNA I VIŠEZNAČNA PRESLIKAVANJA

Korišćenjem pojma kompatibilnosti para preslikavanja neka uopštenja teorema o zajedničkoj nepokretnoj tački iz [2] i [5] su dokazana.

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