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ON CYCLIC (2,m)-GROUPOIDS

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Abstract. Cyclic (2,m)-groupoids represent a generalization of semisymmetric quasigroups. If S is a nonempty set, m a positive integer and F a mapping of S^2 into S^m such that for all $x_1,\dots,x_{m+2} \in S$ $F(x_1,x_2) = (x_3,\dots,x_{m+2})$ implies $F(x_2,x_3) = (x_4,\dots,x_{m+2},x_1)$ then (S,F) is called a cyclic (2,m)-groupoid. Properties of cyclic (2,m)-groupoids and their component operations are determined. It is shown that a class of such (2,m)-groupoids represent an algebraic equivalent of Mendelsohn designs. Some properties of (2,m)-groupoids which are related to some classes of Mendelsohn designs are also given.

1. Introduction

Vector valued groupoids represent a convenient generalization of n-ary groupoids. Various classes of vector valued groupoids were considered in [1], [2], [3], [7]. Some of them are closely related to combinatorial structures and one such class of vector valued groupoids will be considered here.

We shall use the following notation. The sequence $x_p, x_{p+1}, \dots, x_q$ we denote by x_p^q . If $p>q$, then x_p^q will be considered empty.

A groupoid (S,f) is called a left quasigroup if for every $a,b \in S$ the equation $f(x,a) = b$ has a unique solution x, if $f(a,y) = b$ has a unique solution y (S,f) is called a right quasigroup.

A groupoid (S,f) is called idempotent if for all $x \in S$ $f(x,x) = x$.

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Two operations f and g defined on the same set S are said to be orthogonal if for every $a,b \in S$ the system $f(x,y) = a$, $g(x,y) = b$ has a unique solution. That f and g are orthogonal we denote by $f \perp g$. If $f_1,\dots,f_n$ are operations defined on the same set S such that $f_i \perp f_j$ for all $i \neq j$, then $\{f_1,\dots,f_n\}$ is called an orthogonal system.

Let S be a nonempty set, m, n positive integers and F a mapping of S^n into S^m. Then (S,F) is said to be an (n,m)-groupoid (or vector valued groupoid when it is not necessary to emphasize n and m). $|S|$ is called the order of (S,F).

The n-ary operations (n-operations) $f_1,\dots,f_m$ defined by

$$f_i(x_1^n) = y_i \Leftrightarrow (\exists y_1^{i-1}, y_{i+1}^m)\ F(x_1^n) = (y_1^m),\ i=1,\dots,m$$

are called the components operations (or components) of F.

Although every (n,m)-groupoid (S,F) can be interpreted as an algebra $(S,f_1,\dots,f_m)$ with m n-operations, it is often more convenient to consider (n,m)-groupoids in the compact form as an algebra with one (n,m)-operation.

An (n,m)-groupoid (S,F) is called idempotent if for all $x \in S$ $F(x,\dots,x) = (x,\dots,x)$.

An (n,m)-groupoid (S,F) is called cyclic ([11]) if for every $x_1^{n+m} \in S$

$$F(x_1^n) = (x_{n+1}^{n+m}) \Rightarrow F(x_2^{n+1}) = (x_{n+2}^{n+m}, x_1).$$

Cyclic (n,m)-groupoids represent a generalization of cyclic n-groupoids and semisymmetric binary groupoids. For m = 1 a cyclic n-ary quasigroup is obtained (every cyclic n-groupoid is necessarily an n-quasigroup) and if n = 2, m = 1 we get a well known semisymmetric quasigroup (a quasigroup satisfying the identity $y(xy) = x$ is called semisymmetric). Cyclic n-quasigroups were considered in [8] and some of their combinatorial applications in [9], [12]. Some questions concerning general theory of cyclic (n,m)-groupoids were considered in [11], and here we shall consider cyclic (2,m)-groupoids.

2. Cyclic (2,m)-groupoids

Theorem 1. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,\dots,f_m$. If for some $i<j$ $f_i \perp f_j$, then $f_{i+k} \perp f_{j+k}$ for all integers k such that $1-i \le k \le m-i$ and $1-j \le k \le m-j$ and $f_{j-i-p} \perp f_{m+2-p}$ for all integers p such that $j-i-m \le p \le j-i-1$ and $2 \le p \le m+1$.

P r o o f. If $f_i \perp f_j$, then for every $(a,b) \in S^2$ there exist a unique $(x_1^2) \in S^2$ such that $f_i(x_1^2) = a$, $f_j(x_1^2) = b$. Hence there exist unique $(x_3^{i+1}, x_{i+3}^{j+1}, x_{j+3}^{m+2}) \in S^{m-2}$ such that

$$F(x_1^2) = (x_3^{i+1}, a, x_{i+3}^{j+1}, b, x_{j+3}^{m+2})$$

which by the cyclicity of F implies

$$F(x_{m-k+2}^{m-k+3}) = (x_{m-k+4}^{m+2}, x_1^{i+1}, a, x_{i+3}^{j+1}, b, x_{j+3}^{m-k+1}) \qquad (1)$$

and

$$F(x_{i+p+1}^{i+p+2}) = (x_{i+p+3}^{j+1}, b, x_{j+3}^{m+2}, x_1^{i+1}, a, x_{i+3}^{i+p}). \qquad (2)$$

It follows from (1) that $f_{i+k} \perp f_{j+k}$ and from (2) that $f_{j-i-p} \perp f_{m+2-p}$.

Theorem 2. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,...,f_m$ and $i \in N_m$.

1. If f_i is a left quasigroup, then:

a) for $i<m$, $f_k \perp f_{k+i}$, $k=1,...,m-i$,

b) for $i>2$, $f_p \perp f_{m+p-i+2}$, $p=1,...,i-2$,

c) f_{i-1} and f_{m-i+1} are right quasigroups, f_{m-i+2} is a left quasigroup, where all indexes are taken modulo m.

2. If f_i is a right quasigroup, then:

a) for $i<m-1$, $f_k \perp f_{k+i+1}$, $k=1,...,m-i-1$,

b) for $i>1$, $f_p \perp f_{m+p-i+1}$, $p=1,...,i-1$,

c) f_{m-i} is a right quasigroup, f_{i+1} and f_{m-i+1} are left quasigroups, where all indexes are taken modulo m.

P r o o f. 1.a) Let f_i be a left quasigroup. If $(a,b) \in S^2$, then there exists an unique $c \in S$ such that $f_i(c,a)=b$, hence there exist unique $(y_1^{i-1}, y_{i+1}^m) \in S^{m-1}$ such that $F(c,a) = (y_1^{i-1}, b, y_{i+1}^m)$. This implies $F(y_m,c)=(a, y_1^{i-1}, b, y_{i+1}^{m-1})$ and for every $k = 2,...,m-i$

$$F(y_{m-k+1}^{m-k+2}) = (y_{m-k+3}^m, c, a, y_1^{i-1}, b, y_{i+1}^{m-k}).$$

So, for every $(a,b) \in S^2$ the system

$$f_k(x,y) = a, \quad f_{k+i}(x,y) = b$$

has the unique solution x, y, that is, $f_k \perp f_{k+i}$.

Similarly the other cases can be proved.

Corollary 1. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,...,f_m$ and $i \in N_m$. f_i is a quasigroup if and only if f_{i-1} is right quasigroup and f_{i+1} is a left quasigroup, where all indexes are taken modulo m.

Theorem 3. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,...,f_m$ and $i \in N_m$. If f_i is a quasigroup, then for every $k \in N_{m-i-1}$

$$\{f_k, f_{k+i}, f_{k+i+1}\} \text{ and } \{f_k, f_{k+1}, f_{k+i+1}\}$$

are orthogonal systems, and for every $p \in N_{i-2}$

$$\{f_p, f_{m+p-i+1}, f_{m+p-i+2}\} \text{ and } \{f_p, f_{p+1}, f_{m+p-i+2}\}$$

are orthogonal systems.

Proof. By Theorem 2 $f_k \perp f_{k+i}$, $k = 1,\dots,m-i$, and $f_j \perp f_{j+i+1}$, $j = 1,\dots,m-i-1$. Since in every cyclic (n,m)-groupoid every n consequtive components make an orthogonal system ([11]), it follows $f_k \perp f_{k+1}$, $k = 1,\dots,m-1$, hence $\{f_k, f_{k+1}, f_{k+i+1}\}$ is an orthogonal system for all $k = 1,\dots,m-i-1$.

The proof is analogous in other cases.

Theorem 4. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,\dots,f_m$. If for some $i<j$ $f_i \perp f_j$, then f_{j-i} i a left quasigroup and $f_{m-j+i+1}$ is a right quasigroup, moreover, if $j-i>1$, then $f_{m-j+i+2}$ is a left quasigroup and f_{j-i-1} is a right quasigroup.

The proof is similar to the proof of Theorem 2.

Theorems 2 and 4 imply the following corollary.

Corollary 2. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,\dots,f_m$. All components $f_1,\dots,f_m$ are quasigroups if and only if $\{f_1,\dots,f_m\}$ is an orthogonal system.

Corollary 3. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,\dots,f_m$. If for some i,j ($i \neq j$, $i \neq j+1$, $j<m$), $f_i \perp f_j$ and $f_i \perp f_{j+1}$, then f_{j-i} and $f_{m-j+i+1}$ are quasigroups.

Corollary 4. Let (S,F) be a cyclic (2,m)-groupoid with components $f_1,\dots,f_m$. If $f_1 \perp f_k$ for every $k \in \left\{1,\dots,\left[\frac{m+3}{2}\right]\right\}$, then all components $f_1,\dots,f_m$ are quasigroups (where [a] is the greatest integer less or equal to a).

In [11] it was proved that every cyclic (n,m)-groupoid can be defined by an n-groupoid satisfying an identity.

If an infinite sequence of words is defined in the free groupoid (G,f) on n generators $x_1,\dots,x_n$, by

$$w_1(x_1^n) = x_1, \dots, w_n(x_1^n) = x_n,$$

$$w_{i+n}(x_1^n) = f(w_i(x_1^n),\dots,w_{i+n-1}(x_1^n)), \quad i=1,2,\dots,$$

then the following theorem can be proved ([11]).

Let (S,f) be an n-groupoid. (S,f) is an n-groupoid satisfying the identity $w_{n+m+1}(x_1^n) = x_1$ if and only if the (n,m)-groupoid (S,F) defined by $F(x_1^n) = (w_{n+1}(x_1^n),\dots,w_{n+m}(x_1^n))$ is a cyclic (n,m)-groupoid.

Applying the preceding results to cyclic (2,m)-groupoids we get the following.

If (S,F) is a cyclic (2,m)-groupoid with components $f_1,\dots,f_m$, then f_1

satisfies the identity $w_{m+3}(x_1^2) = x_1$ (where $w_i(x_1^2)$ are words obtained as before using a binary operation f_1), and all other components can be expressed by f_1:

$$f_i\,(x_1^2) = w_{i+2}(x_1^2), \quad i = 1,\dots,m.$$

This means that all previously obtained results on cyclic (2,m)-groupoids imply the corresponding properties of groupoids satisfying the identity $w_k(x_1^2) = x_1$.

Some questions concerning groupoids satisfying the identity $w_k(x_1^2) = x_1$ were considered in [4], [5], [6].

3. Mendelsohn designs and cyclic (2,m)-groupoids

Let S be a finite set of v elements. A cyclic k-tuple $\langle a_1^k \rangle$, where a_1^k are distinct elements of S, k≥3, is the following set of ordered pairs:

$$\langle a_1^k \rangle = \{(a_1,a_2), (a_2,a_3), \dots ,(a_{k-1},a_k), (a_k,a_1)\}.$$

A k-Mendelsohn design (kMD) of order v is a pair (S,B) where B is a collection of cyclic k-tuples of elements of S, such that every ordered pair of distinct elements of S belongs to exactly one cyclic k-tuple from B.

In a cyclic k-tuple $\langle a_1^k \rangle$ the pair (a_i,a_{i+t}) is said to be t-apart, where i+t is taken modulo k.

A kMD (S,B) is called r-fold perfect if each ordered pair of elements of S appears t-apart in exactly one cyclic k-tuple from B for all t = 1,...,r. If r = k-1, the design is called perfect.

We note that there is an equivalence between kMDs of order v and decompositions of the complete symmetric directed graph with v vertices K_v^* into arc-disjoint elementary directed circuits of length k.

We shall now prove that kMDs are equivalent to a class cyclic (2,m)-groupoids.

Definition 1. An idempotent cyclic (2,m)-groupoid (S,F) such that $x_1,x_2 \in S$ $x_1 \neq x_2$, $F(x_1^2) = (x_3^{m+2})$ implies that the elements x_1^{m+2} are distinct is called an M-(2,m)-groupoid.

Theorem 5. Every kMD (S,B) of order v defines and is defined by an M-(2,k-2)-groupoid of ordrer v.

P r o o f. Let (S,B) be a kMD of order v. If on S we define a (2,k-2)-operation F such that for every $a \in S$ $F(a,a) = (a,\dots,a)$, and for all $a_1^2 \in S$, $a_1 \neq a_2$,

$$F(a_1^2) = (a_3^k) \Leftrightarrow \langle a_1^k \rangle \in B,$$

then (S,F) is an M-(2,k-2)-groupoid of order v.

Conversely, if an M-(2,k-2)-groupoid (S,F) of order v is given and we define a collection B of cyclic k-tuples by

$$\langle a_1^k \rangle \in B \Leftrightarrow F(a_1^2) = (a_3^k),$$

then it is easy to prove that (S,B) is a kMD of order v.

We have proved that kMDs are equivalent to finite M-(2,k-2)-groupoids. But M-(2,k-2)-groupoids are defined also on infinite sets, hence it follows that M-(2,k-2)-groupoids extend the notion of kMDs to the infinite case.

Theorem 6. If (S,F) is an idempotent cyclic (2,m)-groupoid and $F(x_1^2) = (x_3^{m+2})$, where x_1^{m+2} are not all equal, then $x_i \neq x_{i+1}$, $i = 1,\dots,m+1$, $x_1 \neq x_{m+2}$.

P r o o f. If $F(a,a) = (x_3^{m+2})$, since $F(a,a) = (a,\dots,a)$ it follows $x_i = a$, $i = 3,\dots,m+2$. From $F(x_1,a) = (a,x_4^{m+2})$ it follows $F(a,a) = (x_4^{m+2},x_1)$ and $x_i = a$, $i = 1,4,\dots,m+2$, and similarly in other cases.

Theorem 7. If (S,F) is an idempotent cyclic (2,m)-groupoid such that all its components are quasigroups, then (S,F) is an M-(2,m)-groupoid.

P r o o f. If we assume that

$$F(a,b) = (x_1^m), \quad a \neq b,$$

and for some i, $x_i = a$, then $f_i(a,b) = a$. But f_i must be an idempotent quasigroup, hence we get a contradiction a = b. If two elements among x_1^m are equal, then using the cyclicity of F we get the previous case. So, in the sequence a,b,x_1^m all elements are distinct, that is, (S,F) is an M-(2,m)-groupoid.

The natural question whether the inverse of the preceding theorem is true, that is, are the components of an M-(2,m)-groupoid always quasigroups has a negative answer which follows from an example given in [10].

Definition 2. A cyclic (2,m)-groupoid (S,F) is called r-fold perfect if for every $a,b \in S^2$, and every $t = 2,\dots,r$ there exist unique $x_1^{t-1}, x_t^{m-2} \in S$ such that

$$F(a,x_1) = (x_2^{t-1}, b, x_t^{m-2}).$$

Every cyclic (2,m)-groupoid is 1-fold perfect. If r = m+1 (S,F) is called perfect.

It is clear that r-fold perfect kMDs are equivalent to r-fold perfect M-(2,k-2)-groupoids..

Theorem 8. If a cyclic (2,m)-groupoid (S,F) with components $f_1,\dots,f_m$ is r-fold perfect, then

a) for every $i = 1,\dots,m-r$ and every $t = 1,\dots,r$ $f_i \perp f_{i+t}$,

b) $f_1,\dots,f_{r-1},f_{m-r+2},\dots,f_m$ are quasigroups, f_r is a left quasigroup and f_{m-r+1} is a right quasigroup.

P r o o f. a) Let $a,b \in S$. Since (S,F) is r-fold perfect, for every $t = 1,\dots,r$, there exist unique $x_1^m \in S$ such that

$$F(x_1^2) = (x_3^{i+1}, a, x_{i+2}^{i+t}, b, x_{i+t+1}^m),$$

which means that $f_i \perp f_{i+t}$, $i = 1,...,m-r$.

b) Follows from a) by Theorem 4.

From Theorem 8 and Definition 2 we get the following theorem.

Theorem 9. Let (S,F) be a cyclic (2,m)-groupoid. (S,F) is perfect if and only if all its components are quasigroups.

We note that by Corollary 2 the condition in the preceding theorem - that all components of (S,F) are quasigroups, can be replaced by the condition that components of (S,F) make an orthogonal system.

Theorem 9 and Corollary 2 imply that every perfect cyclic (2,m)-groupoid is a vector valued quasigroup as defined in [1].

If (S_1,F_1) and (S_2,F_2) are two (n,m)-groupoids, then their direct product can be defined in a natural way. If

$$F((x_1,y_1),...,(x_n,y_n)) = ((z_1,u_1),...,(z_m,u_m)) \Leftrightarrow F_1(x_1^n) = (z_1^n) \wedge F_2(y_1^n) = (u_1^n),$$

then $(S_1 \times S_2, F)$ is called the direct product of (S_1,F_1) and (S_2,F_2).

It is not difficult to see that the direct product of two M-(2,m)-groupoids is also an M-(2,m)-groupoid. Hence this direct product can be used to construct new kMDs from the given ones.

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REZIME

Ciklički (2,m)-grupoidi predstavljaju uopštenje polusimetričnih kvazigrupa. Ako je S neprazan skup, m prirodan broj a F preslikavanje S^2 u S^m takvo da za svako $x_1,\dots,x_{m+2} \in S$ iz $F(x_1,x_2) = (x_3,\dots,x_{m+2})$ sledi $F(x_2,x_3) = (x_4,\dots,x_{m+2},x_1)$, onda se (S,F) naziva ciklički (2,m)-grupoid. Određene su neke osobine cikličkih (2,m)-grupoida i njihovih komponentnih operacija. Pokazano je, da jedna klasa takvih (2,m)-grupoida predstavlja algebarski ekvivalent Mendelsonovih sistema. Navedena su i neka svojstva (2,m)-grupoida koji su povezani sa nekim klasama Mendelsonovih sistema.

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