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SOME NEW TYPES OF CONTRACTIONS IN MENGER SPACES

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ABSTRACT

In this paper several fixed point theorems for one or two selfmappings of a Menger space are proved. These mappings are assumed to satisfy some conditions for the diameter of the set of all iterates of these mappings.

1. INTRODUCTION

The notion of a Menger space was introduced in [4] by K. Menger, but the theory of these spaces started to develop after the appearance of the paper [6] of B. Schweizer and A. Sklar in which the general topology of Menger spaces was given. The theory of fixed points of mappings in a Menger space is rapidly developing in the last twenty years. The purpose of this paper is to investigate the existence and uniqueness of a fixed point of one or two selfmappings of a Menger space which satisfy some new types of contraction conserning the orbit.

2. PRELIMINARIES

For the sake of convenience, we first introduce some basic definitions and concepts.

A mapping $F:R\to R^+$ is a distribution function if it is nondecreasing, left-continuous, and $\inf F(t)=0$, $\sup F(t)=1$.

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In the sequel, we allways denote by H the distribution function defined by

$$H(\varepsilon) = \begin{cases} 0, & \varepsilon\le 0 \\ 1, & \varepsilon>0 \end{cases}$$

A commutative, associative and nondecreasing mapping $t: [0,1]\times[0,1] \to [0,1]$ is a T-norm if $t(a,1)=a$ for all $a\in[0,1]$ and $t(0,0)=0$.

Definition 1. *A Menger space is a triplet* (X, F, t), *where* X *is an abstract set of elements,* F *is a mapping of* $X\times X$ *into the set of all distribution functions and* t *is a* T *- norm. We shall denote the distribution function* $F(x,y)$ *by* $F_{x,y}$ *and* $F_{x,y}(\varepsilon)$ *will represent the value of* $F_{x,y}$ *at* $\varepsilon\in R$. *The functions* $F_{x,y}$, $x,y\in X$, *are assumed to satisfy the following conditions:*

1. $F_{x,y}(\varepsilon) = H(\varepsilon)$ *for every* $\varepsilon>0$ *iff* $x=y$
2. $F_{x,y}(0) = 0$, *for all* $x,y\in X$,
3. $F_{x,y}= F_{y,x}$, *for all* $x,y\in X$
4. $F_{x,y}(\varepsilon+\delta)\ge t(F_{x,z}(\varepsilon), F_{z,y}(\delta))$, *for all* $x,y,z\in X$ *and* $\varepsilon,\delta\in R^+$.

The concept of neighbourhoods in a Menger space was introduced by Schweizer and Sklar [6]. If $x\in X$, $\varepsilon>0$ and $\lambda\in(0,1)$, then (ε,λ) - neighbourhood of x, denoted by $U_x(\varepsilon,\lambda)$, is defined by $U_x(\varepsilon,\lambda) = \{y\in X: F_{x,y}(\varepsilon)>1-\lambda\}$.

If $\sup_{a<1} t(a,a)=1$, then (X, F, t) is a Haudorff space in the topology induced by the family of neighbourhoods $\{U_x(\varepsilon,\lambda): x\in X, \varepsilon>0, \lambda\in(0,1)\}$.

The probabilistic diameter of a subset M of X is the mapping D_M defined by

$$D_M(\varepsilon) = \sup_{\delta<\varepsilon} \inf_{x,y\in M} F_{x,y}(\delta).$$

By $\sigma_f(x)$ we shall denote the set $\{f^n x: n\in N_0\}$ by $\sigma_f(x,y)$ the set $\sigma_f(x)\cup\sigma_f(y)$ and by $\sigma_{f,g}(x)$ the set $\{f^n g^k x: n,k\in N_0\}$ (where $N_0=N\cup\{0\}$).

3. FIXED POINT THEOREMS

Throughout this paper let (X, F, t) be a complete Menger space with t-norm such that $\sup_{a<1} t(a,a)=1$ and let $\phi: R^+\to R^+$ be such that

$\lim_{n\to\infty} \phi^n(\varepsilon) = 0$ for all $\varepsilon>1$.

Theorem 1. *Let* $f:X\to X$ *be a continuous mapping such that for all* $x\in X$ *there exists* $m(x)\in N$ *such that for all* $\varepsilon>0$

$$(1) \qquad D_{\sigma_f(x)}(\varepsilon)>1-\varepsilon \Rightarrow D_{\sigma_f(f^{m(x)}x)}(\phi(\varepsilon))>1-\phi(\varepsilon).$$

Then f *has a fixed point and for all* $x_0\in X$ *the sequence of iterates* $\{f^n x_0\}_{n\in N}$ *converges to a fixed point of* f.

Proof. We need only to show that for all $\varepsilon>0$ and all $\lambda\in(0,1)$ there exists $n_0\in N$ such that

$$D_{\sigma_f(f^n x_0)}(\varepsilon)>1-\lambda \qquad \text{for all } n>n_0 .$$

Since $\lim_{n\to\infty} \phi^n(\delta)=0$ for all $\delta>1$, then for all $\varepsilon>0$ and $\lambda\in(0,1)$ there exists $n_0(\varepsilon)\in N$ such that $\phi^{n_0}(1+\varepsilon)<\min\{\varepsilon,\lambda\}$. Further, we form the sequence of natural numbers $\{m(i)\}_{i\in N}$ by

$$m(0)=m(x_0), \quad m(i+1)=m(i)+m(x_{m(i)}) \quad i = 0,1,2,\dots,$$

where $x_k = f^k x_0$, $k\in N$.

From (1), the next implications follow

$$D_{\sigma_f(x_0)}(1+\varepsilon)>1-(1+\varepsilon) \Rightarrow D_{\sigma_f(f^{m(0)}x_0)}(\phi(1+\varepsilon))>1-\phi(1+\varepsilon)\Rightarrow$$

$$\Rightarrow D_{\sigma_f(f^{m(1)}x_0)}(\phi^2(1+\varepsilon))>1-\phi^2(1+\varepsilon) \Rightarrow \dots \Rightarrow$$

$$\Rightarrow D_{\sigma_f(f^{m(n-1)}x_0)}(\phi^n(1+\varepsilon))>1-\phi^n(1+\varepsilon) \Rightarrow$$

$$\Rightarrow D_{\sigma_f(f^n x_0)}(\varepsilon)\geq D_{\sigma_f(f^{m(n-1)}x_0)}(\phi^n(1+\varepsilon))>1-\phi^n(1+\varepsilon)>1-\lambda$$

for all $n>m(n_0-1)$. So, we have proved that for all $x_0\in X$, the sequence $\{f^n x_0\}_{n\in N}$ is a Cauchy sequence and from the completness of the space X, $\lim_{n\to\infty} f^n x_0=x^*$. Since f is a continuous mapping, $fx^*=x^*$, which completes the proof of the theorem.

Theorem 2. *Let* f *be a continuous selfmapping of* (X, F, t) *and let for all* $x,y\in X$ *there exist* $m(x), m(y)\in N$ *such that for all* $\varepsilon>0$

$$D_{\sigma_f(x,y)}(\varepsilon)>1-\varepsilon \Rightarrow D_{\sigma_f(f^{m(x)}x,\, f^{m(y)}y)}(\phi(\varepsilon))>1-\phi(\varepsilon)$$

Then there exists a unique fixed point for f *and for all* $x_0\in X$ *the sequence of iterates* $\{f^n x_0\}_{n\in N}$ *converges to that fixed point.*

Proof. From Theorem 1, the mapping f has a fixed point $x^*=\lim f^n x_0$ where x_0 is any element from X. We shall show that it is a unique fixed point. Let us suppose that $x^*=f^n x^*$ and $y^*=f^n y^*$, for all $n\in N$.

For all $\varepsilon>0$ and all $\lambda\in(0,1)$ there exists $n_0(\varepsilon,\lambda)\in N$ such that $\phi^{n_0}(1+\varepsilon)<\min\{\varepsilon,\lambda\}$. Then, from $D_{\sigma_f(x^*,y^*)}(1+\varepsilon)>1-(1+\varepsilon)$, we get

$$F_{x^*,y^*}(\varepsilon)\geq D_{\sigma_f(x^*,y^*)}(\varepsilon)\geq D_{\sigma_f(x^*,y^*)}(\phi^n(1+\varepsilon))=$$

$$D_{\sigma_f(f^{m(x^*)}x^*,\, f^{m(y^*)}y^*)}(\phi^n(1+\varepsilon))>1-\phi^n(1+\varepsilon)>1-\lambda, \text{ for all } n>n_0(\varepsilon,\lambda),$$

which means that $x^*=y^*$. This completes the proof.

Theorem 3. *Let* f *and* g *be commutative, continuous selfmappings of* (X, F, t) *and let for all* $x\in X$ *there exist* $m(x), k(x)\in N$ *such that for all* $\varepsilon>0$

$$D_{\sigma_{f,g}(x)}(\varepsilon)>1-\varepsilon \Rightarrow D_{\sigma_{f,g}(f^{m(x)}g^{k(x)}x)}(\phi(\varepsilon))>1-\phi(\varepsilon)$$

Then for each $x_0\in X$ *the sequence* $\{f^n g^n x_0\}_{n\in N_0}$ *konverges to some common fixed point* $x^*\in X$ *of* f *and* g.

Proof. We can forme a sequence $\{x_i\}_{i\in N_0}$ by $x_{i+1}=f^{m(x_i)}g^{k(x_i)}x_i$. Then we have

$$D_{\sigma_{f,g}(x_0)}(1+\varepsilon)>1-(1+\varepsilon) \Rightarrow D_{\sigma_{f,g}(x_1)}(\phi(1+\varepsilon))>1-\phi(1+\varepsilon) \Rightarrow \dots$$

$$\Rightarrow D_{\sigma_{f,g}(x_n)}(\phi^n(1+\varepsilon))>1-\phi^n(1+\varepsilon).$$

For all $\varepsilon>0$ and all $\lambda\in(0,1)$ there exists $n_0(\varepsilon,\lambda)\in N$ such that $\phi^n(1+\varepsilon)<\min\{\varepsilon,\lambda\}$ for all $n>n_0$. If $n>n_0$ we get

$$D_{\sigma_{f,g}(x_n)}(\varepsilon) \geq D_{\sigma_{f,g}(x_n)}(\phi^n(1+\varepsilon)) > 1-\phi^n(1+\varepsilon) > 1-\lambda$$

which implies that all subsequences of $\{f^n g^k x_o\}_{n,k\in N_o}$ in which the indexes n and k converge to ∞, are Cauchy sequences and $\lim\limits_{\substack{n\to\infty \\ k\to\infty}} f^n g^k x_o = x^* \in X$. The following three subsequences converge to x^*

$$\{f^n g^n x_o\}_{n\in N_o}, \quad \{f(f^n g^n)x_o\}_{n\in N}, \quad \{g(f^n g^n)x_o\}_{n\in N_o}.$$

By the continuity of f and g we obtain that

$$x^* = fx^* = gx^*$$

which means that x^* is a common fixed point of f and g.

Theorem 4. *Let* f *and* g *be selfmappings of* (X, F, t) *and let for all* $x,y \in X$ *there exist* $\{m(x), k(x), m(y), k(y)\}$ N *such that for all* $\varepsilon > o$.

$$D_{\sigma_{f,g}(x)\cup\sigma_{f,g}(y)}(\varepsilon) > 1-\varepsilon \Rightarrow D_{\sigma_{f,g}(f^{m(x)}g^{k(x)}x)\cup\sigma_{f,g}(f^{m(y)}g^{k(y)}y)}(\Phi(\varepsilon)) > 1-\Phi(\varepsilon)$$

Then there exists a unique common fixed point for f *and* g. The proof The proof of this theorem is similar to the proof of Theorem 2 and Theorem 3, so it is omited.

4. CONNECTION WITH METRIC SPACES

We shall consider a Menger space (X, F, t). The function $d: X\times X \to R^+$ defined by

$$d(x,y) = \inf_{\varepsilon>0} \{\varepsilon+1-F_{x,y}(\varepsilon)\}$$

is a metric in X. In [7] it was proved that, if $\sup\limits_{a>1} t(a,a)=1$ and $t \geq t_m$, then the metric d induces the same topology which is induced by (ε,λ)-neighbourhoods.

The following theorem is due to P.R. Meyers and it is used for the proof of Theorem 6.

Theorem 5. [5] *Let* (X,d) *be a complete metric space,* $f: X \to X$ *a continuous mapping and the following conditions are satisfied*

1. f *has a unique fixed point* x^*.
2. *For each* $x \in X$ *the sequence* $\{f^n x\}_{n\in N}$ *converges to* x^*.
3. *There exists an open neighbourhood* U *of* x^* *with the property that for any given open set* V *including* x^* *there*

is an $n_0 \in N$ *such that* $n > n_0$ *implies* $f^n(U) \subseteq V$.

Then for each $k \in (0,1)$ *there exists a metric* d_k, *topologically equivalent to the metric* d *so that*

$$d_k(fx,fy) \leq k\, d_k(x,y), \quad x,y \in X.$$

Theorem 6. *If all conditions of Theorem 2 are satisfied, then for every* $k \in (0,1)$ *there exists the metric* d_k *topologically equivalent to a metric* d *which induces the* (ε,λ)*- uniformity such that*

$$d_k(fx,fy) \leq k\, d_k(x,y) \text{ for all } x,y \in X.$$

Proof: By Theorem 2 the mapping f satisfies conditions 1 and 2 from Theorem 5. In order to prove condition 3 we shall take that $U=X$ and $V=\{y:\ y \in X,\ F_{x^*,y}(\varepsilon) > 1-\lambda\}$, $\varepsilon > 0$, $\lambda \in (0,1)$. Since $\{f^n\}_{n \in N}$ is a Cauchy sequence converging to the fixed point and since for all $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $n_0(\varepsilon,\lambda)$ such that $\phi^n(1+\varepsilon) < \min\{\varepsilon,\lambda\}$ for all $n > n_0$, we have (using the same argument as in Theorem 1).

$$D_{\sigma_f(x^*,f^n x)}(\varepsilon) \geq D_{\sigma_f(x^*,f^{m(n_0-1)}x)}(\phi^{n_0}(1+\varepsilon)) > 1-\phi^{n_0}(1+\varepsilon) > 1-\lambda$$

for all $n > m(n_0-1)$ where the sequence of numbers $\{m(i)\}_{i \in N}$ is the same as in Theorem 1.

This completes the proof of the theorem.

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Rezime

NEKI NOVI TIPOVI KONTRAKCIJA U MENGEROVIM PROSTORIMA

U ovom radu je dokazano nekoliko teorema o nepokretnoj tački za jedno ili dva preslikavanja nekog Mengerovog prostora u samog sebe. Preslikavanja su takva da zadovoljavaju neke uslove vezana za dijametar orbite.

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