Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 18,1,57-67(1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

ON SOME OPERATIONS ON NORMAL DIGRAPHS

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ABSTRACT

In the paper the necessary and sufficient, sometimes only sufficient, conditions are given for some operations on digraphs to be a normal digraph. Some related results are obtained.

Throughout this paper (except Definition 2), we shall consider digraphs without multiple arcs and loops i.e. digraphs in the sense [1] or [3]. The terminology and notation are as in [1] or [3]. If D is a given digraph, then A(D), V(D) and E(D) denote the adjacency matrix, the set of vertices and the set of arcs of digraph D, respectively. In [5] the following definition is introduced.

Definition 1. *A digraph is called normal if its adjacency matrix is a normal one.*

A vertex w of a digraph D is called a common predecessor of any two vertices u and v of D, if $(w,u) \in E(d)$ and $(w,v) \in E(D)$. A common successor of vertices u and v is defined

AMS Mathematics Subject Classification (1980): 05C20.

Key words and phrases: Digraph, operations on digraphs, normal digraph.

analogously. It can be easily shown [5] that a digraph D is a normal one if and only if for each two vertices of D, the number of common predecessors is equal to the number of common successors. This means that for each vertex u of a normal digraph D, the indegree $id_D(u)$ of u is equal to the outdegree $od_D(u)$ of u.

Let D and G be two digraphs on the same set of vertices and let u and v be any two of their vertices. We denote by $prc_D(u,v)$ (more precisely $prc_{E(D)}(u,v)$) the number of common predecessors w of vertices u and v in D such that $(w,u)\in E(D)$ and $(w,v)\in E(D)$ and by $prc_{D,G}(u,v)$ the number of common predecessors w of vertices u and v such that $(w,u)\in E(D)$ and $(w,v)\in E(G)$. Similarly, by $suc_D(u,v)$ and $suc_{D,G}(u,v)$, we denote the corresponding number of successors. When there is no danger of confusion, we shall omit D in $prc_D(u,v)$ as in $suc_D(u,v)$.

If D is a given digraph, then we call a complement of D a digraph $\bar{D}$ which has V(D) as its vertex set, but vertex u is adjacent to vertex v in $\bar{D}$ if and only if u is not adjacent to v in D.

Theorem 1. *The complement* $\bar{D}$ *of a normal digraph* D *is a normal digraph.*

Proof. If $A(D) = A$ then $A(\bar{D}) = J-A-I$, where J is a square matrix all of whose elements are equal to 1 and I is the unit matrix. It is easy to check that $A(\bar{D})$ is a normal matrix if A is a normal one.

Theorem 2. *Let* D_1 *and* D_2, $E(D_1)\cap E(D_2) = \emptyset$ *be two normal digraphs. The union* D *of digraphs* D_1 *and* D_2 *is a normal digraph if for each two vertices* u *and* v *of* D *the relation* $prc_{D_1,D_2}(u,v)+prc_{D_2,D_1}(u,v) = suc_{D_1,D_2}(u,v)+suc_{D_2,D_1}(u,v)$ *holds.*

Proof. Let $D_1 = (V_1,E_1)$ and $D_2 = (V_2,E_2)$ and let $D_1' = (V_1\cup V_2,E_1)$ and $D_2' = (V_1\cup V_2,E_2)$ and let $A(D_1') = A_1$ and

$A(D_2') = A_2$. Then $A(D) = A_1+A_2$ and $AA^T = A^TA$ if $A_2A_1^T+A_1A_2^T =$ $= A_2^TA_1+A_1^TA_2$ which proves the theorem.

Corollary 1. *The union of a finite number of digraphs which have disjoint vertex sets is a normal digraph if and only if all of its components are normal digraphs.*

Proof is obvious.

Example. A cycle is a normal digraph [5], and using Corollary 1 and Theorem 1, we get that digraph D obtained from a complete symmetric digraph K_p by deleting all the arcs of a cycle or union of vertex-disjoint cycles is a normal digraph.

Let D_1 and D_2 be two vertex-disjoint digraphs. The join of the digraphs D_1 and D_2 is a digraph obtained from the union $D_1 \cup D_2$ by adding all the arcs (u,v) whenever $u \in V(D_1)$ and $v \in V(D_2)$ or $u \in V(D_2)$ and $v \in V(D_1)$.

Theorem 3. *The join of digraphs D_1 and D_2 is a normal digraph if and only if digraphs D_1 and D_2 are normal ones.*

Proof. Let D be the join of the digraphs D_1 and D_2 and let $|V(D_1)| = p$ and $|V(D_2)| = q$.

Suppose D is a normal digraph. For each vertex u from $V(D_1)$ the $id_{D_1}(u) = id_D(u)-q$ and $od_{D_1}(u) = od_D(u)-q$ and so $id_{D_1}(u) = od_{D_1}(u)$ holds. Similarly, it holds if u is from $V(D_2)$. Furthermore, for any two vertices u and v both from the same set of vertices $V(D_1)$ or $V(D_2)$ we can say, without loss of generality, from V_1, the $prc_D(u,v) = prc_{D_1}(u,v)+q$ holds since $suc_D(u,v) = suc_{D_1}(u,v)+q$, and, consequently $prc_{D_1}(u,v) =$ $suc_{D_1}(u,v)$ holds. Since, D_1 (as D_2) is a normal digraph.

To see the sufficiency, let D_1 and D_2 be normal digraphs. Let u and v be any two vertices of D. If u and v are both from $V(D_1)$ or from $V(D_2)$ similarly, as above, we see that $id(u)=od(u)$ and $prc(u,v)=suc(u,v)$. But if one vertex, say u, is from $V(D_1)$ and other, say v, is from $V(D_2)$, then obviously $id(u)=od(u)$ and $id(v)=od(v)$ holds and the number of common predecessors of u and v which belong to $V(D_1)$ is equal to the $id_{D_1}(u)$ and the number of common predecessors of u and v which belong to $V(D_2)$ is equal to the $id_{D_2}(v)$ since the number of common predecessors of u and v on the whole is equal to the $id_{D_1}(u) + id_{D_2}(v)$. Analogously, the number of common successors of u and v is equal to the $od_{D_1}(u) + od_{D_2}(v)$. Then $prc(u,v) = suc(u,v)$ holds.

This completes the proof of the theorem.

Let us call a dijoin of digraphs D_1 and D_2 a digraph D obtained from $D_1 \cup D_2$ if each of the vertices of D_1 is joined by exactly one arc arbitrarily directed with each of the vertices of D_2.

Theorem 4. *The dijoin* D *of normal digraphs* D_1 *and* D_2 *is a normal digraph if the following conditions are satisfied:*

(i) $|V(D_1)| = 2r$, $|V(D_2)| = 2s$ *(*r,s *are natural numbers) and each of the vertices of* D_1 *is adjacent to exactly* s *vertices of* D_2 *and each of the vertices of* D_2 *is adjacent to exactly* r *vertices of* D_1.

(ii) For each two vertices u *and* v *of* D *the following holds*
$$prc_{D_1,E}(u,v) + prc_{E,D_2}(u,v) = suc_{D_1,E}(u,v) + suc_{E,D_2}(u,v),$$
where E *is the set of new arcs added to the union* $D_1 \cup D_2$ *to be the dijoin of* D_1 *and* D_2.

Proof. Let u and v be any two vertices belonging to $V(D_1)$ both. Then, $prc_D(u,v) = prc_{D_1}(u,v) + prc_E(u,v)$ holds and

$suc_D(u,v) = suc_{D_1}(u,v) + suc_E(u,v)$ also. There is for each pair u,v a partition of the set of vertices of D_2 in four classes: $V_1 = \{w|(w,u)\in E$ and $(w,v)\in E\}$, $V_2 = \{w|(u,w)\in E$ and $(v,w)\in E\}$, $V_3 = \{w|(u,w)\in E$ and $(w,v)\in E\}$ and $V_4 = \{w|(w,u)\in E$ and $(v,w)\in E\}$. If $id_E(w)$ (similarly $od_E(w)$) denote the number of arcs in E incident to vertex w, then $od_E(u) = |V_2|+|V_3|$, $id_E(u) = |V_1|+$ $+|V_4|$, $od_E(v) = |V_2|+|V_4|$ and $id_E(v) = |V_1|+|V_3|$ holds and by $od_E(u) = id_E(u) = od_E(v) = id_E(v)$, it follows that $|V_1| = |V_2|$ i.e. $prc_E(u,v) = suc_E(u,v)$ and finally accounting $prc_{D_1}(u,v) =$ $suc_{D_1}(u,v)$ it follows that $prc(u,v) = suc(u,v)$. In a similar way, we get the same conclusion if $u,v\in V(D_2)$.

But, if $u\in V(D_1)$ and $v\in V(D_2)$, then they can have a common predecessor and a common successor in $V(D_1)$ and in $V(D_2)$ and these vertices satisfy condition (ii) of the theorem.

This completes the proof of the theorem.

Corollary 2. *The bipartite tournament with bipartition* (V_1,V_2) *is a normal digraph if and only if* $|V_1| = 2r$, $|V_2| = 2s$ *(r,s are natural numbers) and each of the vertices from* V_1 *(from* V_2*) is adjacent to exactly* s (r) *vertices from* V_2 (V_1).

Proof. Follows by Theorem 4 if we notice that condition (ii) is trivially satisfied.

The subdivision of digraph D is a digraph S(D) obtained from D by inserting a new vertex into every arc of D and introducing the induced orientation in the new arcs. The line digraph L(D) of a digraph D has the arcs of the given digraph D as its vertices, and x is adjacent to y in L(D) whenever arcs x,y induce a walk in D.

Theorem 5. *The subdivision* S(D) *of a digraph* D, *which do not have isolate vertices, is a normal digraph if and only* *if* D *is a cycle or union of vertex-disjoint cycles.*

Proof. If D is a cycle C_p, then S(D) = C_{2p} is a normal digraph.

Let S(D) be a normal digraph with V(S(D)) = V∪S, where V is the set of vertices of D and S is the set of new vertices added to D making the subdivision of D. No vertices u,v∈V or u∈V and v∈S can have a common predecessor or successor which can be easily shown. Let u,v∈S. If w is their common predecessor (there can be at most one) and ν their common successor (at most one), then from w to ν(w,ν∈V) two arcs lead in D which is impossible. On other hand, if from any vertex of D there lead at least two arcs to any different vertices, then in S(D) there exist two vertices which have a common predecessor and do not have a common successor. Analogously, if to any vertex of D there lead at least two arcs, then in S(G) there exist two vertices which have a common successor and do not have a common predecessor. Therefore, for each vertex w of D od(w) = id(w)=1 holds, which completes the proof of the theorem.

Theorem 6. *The line digraph* L(D) *of a digraph* D, *which contains in each of its connected components at least two arcs which induce a walk in* D, *is a normal digraph if and only if* D *is a cycle or the union of vertex-disjoint cycles.*

Proof. If D is a cycle C_p, then L(D)=C_p is a normal digraph.

Let L(D) be a normal digraph. Any two vertices x and y of L(D) can not have a common predecessor and a common sucessor simultaneously, because then x and y are multiple arcs in D. Therefore, in D there cannot exist a vertex u such that id(u)$\geq$1 and od(u)$\geq$2 or id(u)$\geq$2 and od(u)$\geq$1, because then in L(D) there exist a pair of vertices which have a common predecessor and do not have a common successor or vice versa.

So, if for any vertex u of D $id(u)\geq 1$ and $od(u)\geq 1$, then $id(u) = od(u)=1$ holds.

Furthermore, if, for example, for any vertex u of D $id(u)=0$ and $od(u)\geq 1$, then each arc x of D which leads from u terminates in a vertex of D which has an outdegree equal to zero ($id(x) = od(x)$ since L(D) is a normal digraph) and, consequently, digraph D contains a component which has no two arcs which induce a walk in D.

This completes a proof of the theorem.

Let B be a set of n-tuples $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of symbols 1, 0 and −1 which does not contain n-tuple $(0,0,\ldots,0)$. If D is a (pseudo)-digraph with at most p parallel arcs between any two vertices or loops of a vertex in D, then $\bar{D}$ is a digraph which have the same set of vertices as D and for any two vertices u,v of D (if loops are not allowed, then $u\neq v$) from u to v leads p-d arcs, where d is the number of arcs leading from u to v in D.

The following definition is an extension of the definition from [4] of the generalized direct product of graphs to (pseudo)-digraphs.

Definition 2. *The generalized direct product with a basis* B *of digraphs* $D_1, D_2, \ldots, D_n$ *is the digraph* $D=GDP(B, D_1, D_2, \ldots, D_n)$ *whose set of vertices is the Cartesian product of the sets of vertices of digraphs* $D_1, D_2, \ldots, D_n$.

For the two vertices $u=(u_1, u_2, \ldots, u_n)$ *and* $v=(v_1, v_2, \ldots, v_n)$ *of* D *constructs all the possible arc selection of the following type. For each* $\beta \in B$ *and for any* i $(i=1,2,\ldots,n)$ *select an arc from* u_i *to* v_i *in* D_i *if* $\beta_i=1$, *an arc from* u_i *to* v_i *in* $\bar{D}_i$ *if* $\beta_i=-1$ *and suppose* $u_i=v_i$ *if* $\beta_i=0$. *The number of arcs going from* u *to* v *is equal to the number of such selections.*

If B consists of the n-tuples of symbols 1 and 0, the operation is called a non-complete extended p-sum (NEPS) and is defined in [2]. This operation contains the well known operations such as sum, product, strong product, p-sum, and so on. Further, if for example, in the previous definition n=2 and $B = \{(1,1), (1,0), (1,-1), (0,1)\}$ the resulting operation is called the lexicographic product (or composition).

A digraph is called a regular of degree r, if each indegree and each outdegree equals r.

Lemma 1. *If* D *is a regular normal digraph and* $\bar{D}$ *is its complement, then* $suc_{D,\bar{D}}(u,v) = prc_{\bar{D},D}(u,v)$ *holds, for any two vertices* u *and* v *of* D.

Proof. Since for a regular normal digraph D, $A(D)\cdot A(\bar{D})^T = A(\bar{D})^T\cdot A(D)$ holds, the statement of this Lemma follows immediately.

Theorem 7. *The generalized directed product* D *with a basis* B *of normal digraphs* $D_1, D_2, \dots, D_n$ *is a normal digraph if one of the following conditions is satisfied:*

(i) D_i, $i=1,2,\dots,n$ *are the regular normal digraphs.*

(ii) B *does not contain an* n*-tuple whose any term is* -1.

(iii) There are no two n*-tuples in* B *which have, for at least one* $i, 1\le i\le n$, *i-th term* 1 *in one of the* n*-tuple and* -1 *in another one.*

Proof (i) Let for any pair of n-tuples $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ and $\beta' = (\beta'_1, \beta'_2, \dots, \beta'_n)$ from B the sets $I_j \subset \{1,2,\dots,n\}$, $j=1,2,\dots,9$ be defined as follows: $I_1 = \{r \mid \beta_r = 1 \text{ and } \beta'_r = 1\}$, $I_2 = \{r \mid \beta_r = 1 \text{ and } \beta'_r = 0\}$, $I_3 = \{r \mid \beta_r = 1 \text{ and } \beta'_r = -1\}$, $I_4 = \{r \mid \beta_r = 0 \text{ and } \beta'_r = 1\}$, $I_5 = \{r \mid \beta_r = 0 \text{ and } \beta'_r = -1\}$,

$I_6 = \{r|\beta_r = -1 \text{ and } \beta'_r = 1\}$, $I_7 = \{r|\beta_r = -1 \text{ and } \beta'_r = 0\}$,
$I_8 = \{r|\beta_r = -1 \text{ and } \beta'_r = -1\}$ and $I_9 = \{r|\beta_r = 0 \text{ and } \beta'_r = 0\}$.

Let $u = (u_1,u_2,\dots,u_n)$ and $v = (v_1,v_2,\dots,v_n)$ be any two vertices of D and let $w = (w_1,w_2,\dots,w_n)$ be their common predecessor. This means that there exist a pair of n-tuples $\beta = (\beta_1,\beta_2,\dots,\beta_n)$ and $\beta' = (\beta'_1,\beta'_2,\dots,\beta'_n)$ in B such that w_i is adjacent to u_i and to v_i in D_i if $i\in I_1$, v_i is adjacent to u_i in D_i if $i\in I_2$, w_i is adjacent to u_i in D_i and to v_i in $\bar{D}_i$ if $i\in I_3$, u_i is adjacent to v_i in D_i if $i\in I_4$, u_i is adjacent to v_i in $\bar{D}_i$ if $i\in I_5$, w_i is adjacent to u_i in $\bar{D}_i$ and to v_i in D_i if $i\in I_6$, v_i is adjacent to u_i in $\bar{D}_i$ if $i\in I_7$, w_i is adjacent to u_i and to v_i in $\bar{D}_i$ if $i\in I_8$ and $u_i=v_i$ if $i\in I_9$. The number of such predecessors (if any) for any pair of n-tuples $\beta,\beta'\in B$ is equal to

$$(1)\qquad \prod_{i\in I_1} \mathrm{prc}_{D_i}(u_i,v_i) \times \prod_{i\in I_3} \mathrm{prc}_{D_i,\bar{D}_i}(u_i,v_i) \times$$

$$\times \prod_{i\in I_6} \mathrm{prc}_{\bar{D}_i,D_i}(u_i,v_i) \times \prod_{i\in I_8} \mathrm{prc}_{\bar{D}_i}(u_i,v_i).$$

If there exists a common predecessor of u and v for any pair of n-tuples, then, by Lemma 1, for the same pair of n-tuples there exists a common successor of u and v (now β' defines the successor of u and β defines the successor of v) and vice versa.

Let $w' = (w'_1,w'_2,\dots,w'_n)$ be a such successor corresponding to the predecessor w. Then u_i and v_i are adjacent to w'_i in D_i if $i\in I_1$, u_i is adjacent to v_i in D_i if $i\in I_4$, u_i is adjacent to w'_i in D_i and v_i is adjacent to w'_i in $\bar{D}_i$ if $i\in I_6$, v_i is adjacent to u_i in D_i if $i\in I_2$. v_i is adjacent to u_i in $\bar{D}_i$ if $i\in I_7$, u_i is adjacent to w'_i in $\bar{D}_i$ and v_i is adjacent to w_i in D_i if $i\in I_3$, u_i is adjacent to v_i in $\bar{D}_i$ if $i\in I_5$, u_i and v_i are adjacent to w_i in $\bar{D}_i$ if $i\in I_8$ and $u_i=v_i$ if $i\in I_9$. The number of such successors (if any) is equal to

$$(2)\qquad \prod_{i\in I_1} suc_{D_i}(u_i,v_i) \times \prod_{i\in I_6} suc_{D_i,\bar{D}_i}(u_i,v_i) \times$$

$$\times \prod_{i\in I_3} suc_{\bar{D}_i,D_i}(u_i,v_i) \times \prod_{i\in I_8} suc_{\bar{D}_i}(u_i,v_i).$$

If any one of the sets I_j is empty, then the corresponding products in (1) and (2) should be omitted.

Since, D_i and $\bar{D}_i$, $i=1,2,\dots,n$ are the normal digraphs using Lemma 1, we get that (1) is equal to (2). Adding up quantities (1), respectively (2) for each pair $\beta,\beta'\in B$, we get the statement (i).

(ii) Follows similarly as in (i), if we take into account that the sets I_3,I_5,I_6,I_7 and I_8 are empty.

(iii) Follows similarly as in (i), if we take into account that the sets I_3 and I_6 are empty.

This completes the proof of the theorem.

This theorem holds also if the digraphs D_i, $i=1,2,\dots$ $\dots,n$ have multiple arcs and/or loops.

Theorem 8. *The lexicographic product* $D_1[D_2]$ *of digraphs* D_1 *and* D_2 *is a normal digraph, if* D_1 *is a normal digraph and* D_2 *is a regular normal digraph.*

Proof. The adjacency matrix A of lexicographic product $D_1[D_2]$ is $A = (A(D_1)\otimes J)+(I\otimes A(D_2))$, where $X\otimes Y$ denote the Kronecker product of matrices X and Y and J and I have the same meaning as in Theorem 1. It is easily to check that $A\cdot A^T=$ $=A^T\cdot A$, which proves the Theorem.

Finally I wish to thank Aleksandar Torgašev for making his unpublished manuscript [5] available to me.

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REZIME

O NEKIM OPERACIJAMA NA NORMALNIM DIGRAFOVIMA

U radu su dati potrebni i dovoljni, neki put samo dovoljni, uslovi za neke operacije na digrafovima da bi rezultujući digraf bio normalan digraf.

Received by the editors June 1, 1986.