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A SECOND ORDER UNIFORM NUMERICAL METHOD FOR A TURNING POINT PROBLEM

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ABSTRACT

A singularly perturbed second order boundary value problem with a turning point is considered. A non-equidistant generalization of the Gushchin-Schennikov scheme is used on a special discretization mesh and the second order convergence uniform in a small perturbation parameter is proved.

1. INTRODUCTION

The non-equidistant generalization of the Gushchin--Shchennikov scheme [1] was introduced in [2], where we considered the numerical solution of a singularly perturbed boundary value problem without turning points. The reason for using such a finite-difference scheme was explained in [2]. Here, we shall use a scheme of the same class to solve the following turning point problem numerically:

(1) $$-\varepsilon^2 u''-xa(x)u'+c(x)u=f(x),\ x\in I=[0,1],\ u(0)=u(1)=0,$$

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where ε denotes a small perturbation parameter, $0<\varepsilon<1$; $a\in C^2(I)$ and c, $f\in C^3(I)$ are given functions and

$$a(x) \geq a_* > 0, \quad c(x) > c_* > 0, \quad x\in I.$$

(We shall consider interval $I = [0,1]$ only, although the case when $I = [-1,1]$ can be treated analogously, cf. [3]).

It is well known that the solution $u_\varepsilon\in C^4(I)$ to problem (1) exists uniquely. Its derivatives up to the third order were estimated in [3] in the case when $a\in C^1(I)$ and c, $f\in C^2(I)$. Using the same technique we can prove here:

Lemma 1. *Let* $u_\varepsilon\in C^4(I)$ *be the solution to problem (1). Then the following estimates hold:*

$$|u_\varepsilon^{(i)}(x)| \leq \begin{cases} M\varepsilon^{-i}, & 0 \leq x \leq m_0\varepsilon & \text{(2a)} \\ M(\varepsilon^{-i}\exp(-m_1x/\varepsilon)+x^{p-i}), & m_0\varepsilon \leq x \leq 1 & \text{(2b)} \end{cases}$$

$$i=1,2,3$$

$$|u_\varepsilon^{(i)}(x)| \leq \begin{cases} M\varepsilon^{-i}, & 0 \leq x \leq m_0\varepsilon & \text{(3a)} \\ M(\varepsilon^{-i}\exp(-m_1x/\varepsilon)+\varepsilon^{-2}x^{p+2-i}, & m_0\varepsilon\leq x\leq 1 & \text{(3b)} \end{cases}$$

$$i=3,4$$

where $p = \min(1, c_*/a(0))$, m_0 *and* m_1 *are arbitrary positive constants and* M *is independent of* ε. □

These estimates will be used in Section 4 in the proof of the consistency uniform in ε. The uniform consistency is due to the use of a special non-equidistant discretization mesh which is dense near the origin. The mesh is given in Section 2 by a mesh generating function which is the function from [2], modified analogously to [3]. The stability uniform in ε and the second order accuracy are obtained because of the use of the non-equidistant Gushichin-Shchennikov scheme which is given in Section 3. The paper ends with Section 5 where some numerical results are presented.

Besides in [3], problem (1) was considered in some earlier papers, such as [4], [5], [6], [7], but the conditions were less general there. Note that the highest uniform convergence order obtained in papers [3]-[7] was 1.

Throughout the paper we let $p=\min(1, c_*/a(0))$ and denote by M any positive constant bounded independently of ε and of the discretization mesh.

2. THE MESH

Let $\lambda(t) = \lambda_1(t)^{2/p}$, $t\in I$,

$$\lambda_1(t) = \begin{cases} \psi(t) := A\varepsilon^{p/2}t/(q-t), & t\in[0,\tau] \\ \pi(t) := \psi'(\tau)(t-\tau)+\psi(\tau), & t\in[\tau,1] \end{cases},$$

where $q\in(0,1)$, $A\in(0,q/\varepsilon^{p/2})$ are fixed numbers and $(\tau, \psi(\tau))$ is the contact point of the tangent line from (1,1) to curve $\psi(t)$. We can easily get that $\tau\in(0, q)$ exists uniquely and that $q-\tau = M\varepsilon^{p/4}$.

Function $\lambda(t)$ has the following properties:

(4a) $\lambda^{(k)}(t) \geq 0$, $k=1,2,3$, $t\in I$,

(4b) $\lambda'(t) \leq M\lambda(t)^{1-p/2}$, $t\in I$,

(4c) $\lambda''(t) \leq M\lambda(t)^{1-p}$, $\tau \leq t \leq 1$.

The discretization mesh I_h is generated by $\lambda(t)$, i.e. the mesh points are given by

$$x_i = \lambda(t_i), \quad t_i = ih, \quad i=0,1,\dots,n, \quad h=1/n, \ n\in N.$$

Let $h_i = x_i-x_{i-1}$, $i=1,2,\dots,n$, and $x_{i+1/2} = x_i+h_{i+1}/2$.

3. THE SCHEME

Let $\{w_i\}$ be a mesh function on I_h. We introduce the following discrete operators:

$$D_c''w_i = (2h_{i+1}w_{i-1}-(h_i+h_{i+1})w_i+h_iw_{i+1})/(h_ih_{i+1}(h_i+h_{i+1})),$$

$$D_c'w_i = (w_{i+1}-w_{i-1})/(h_i+h_{i+1}),$$

$$D_+'w_i = (w_{i+1}-w_i)/h_{i+1},$$

$$D_M''w_{i+1/2} = \alpha_iw_{i-1}+\beta_iw_i+\gamma_iw_{i+1}+\delta_iw_{i+2},$$

$$D_M'w_{i+1/2} = (w_{i+1}-w_i)/h_{i+1},$$

$$D_M^ow_{i+1/2} = (3w_i-w_{i-1})/2,$$

where the notation $w_{i+1/2}$ should be understood formally, and:

$$\alpha_i = (2h_{i+2}+h_{i+1})/(h_i(h_i+h_{i+1})(h_i+h_{i+1}+h_{i+2})),$$

$$\beta_i = -(2(h_{i+2}-h_i)+h_{i+1})/(h_ih_{i+1}(h_{i+1}+h_{i+2})),$$

$$\gamma_i = (2(h_{i+2}-h_i)-h_{i+1})/(h_{i+1}h_{i+2}(h_i+h_{i+1})),$$

$$\delta_i = (2h_i+h_{i+1})/(h_{i+2}(h_{i+1}+h_{i+2})(h_i+h_{i+1}+h_{i+2})).$$

Before forming the discretization of problem (1), it is convenient to rewrite the problem in the following form:

$$Lu := -\varepsilon^2 u'' - a(x)(xu)' + (a(x)+c(x))u = f(x), \quad x \in I,$$

(5)

$$u(0) = u(1) = 0.$$

We shall use the following schemes for problem (5):

- central scheme

$$L_c^h w_i := -\varepsilon^2 D_c'' w_i - a(x_i) D_c' x_i w_i + (a+c)(x_i) w_i,$$

- mid-point scheme

$$L_M^h w_{i+1/2} := -\varepsilon^2 D_M'' w_{i+1/2} - a(x_{i+1/2}) D_M' x_{i+1/2} w_{i+1/2} +$$

$$+ (a+c)(x_{i+1/2}) D_M^o w_{i+1/2},$$

- up-wind scheme

$$L_u^h w_i := -\varepsilon^2 D_c'' w_i - a(x_i) D_+' x_i w_i + (a+c)(x_i) w_i.$$

On mesh I_h we form the discretization of problem (5):

$$w_o = 0$$

(6)

$$L_G^h w_i := \begin{cases} L_c^h w_i = f(x_i), & \text{if} \quad \rho_i := h_i x_{i-1} a(x_i)/(2\varepsilon^2) \le 1 \\ L_M^h w_{i+1/2} = f(x_{i+1/2}), & \text{if} \quad \rho_i > 1, \end{cases}$$

$$i = 1, 2, \ldots, n-2,$$

$$L_G^h w_{n-1} := L_u^h w_{n-1} = f(x_{n-1}),$$

$$w_n = 0.$$

This is the non-equidistant generalization of the Gushchin - Shchennikov scheme, similar to the one from [2].

From now on we shall take

$$n \geq 2\lambda'(1)a_1/a_*, \tag{7}$$

where $a'(x) \geq -a_1$, $x \in I$, $a_1 \geq 0$. Inequality (7) implies that

$$-\varepsilon^2\gamma_i - a(x_{i+1})x_{i+1}/h_{i+1} \leq 0 \quad \text{when} \quad \rho_i > 1,$$

thus it follows that the matrix, corresponding to system (6) is an M-matrix, cf. [2]. Then the stability uniform in ε is immediate.

Schemes L_C^h and L_M^h are second order accurate and L_u^h is a first order scheme. In spite of the use of L_u^h at point x_{n-1}, we can prove that there is no loss of accuracy. The technique for this is the same as in [2] and we shall not repeat the proof here. We shall only prove the consistency uniform in ε (see the next Section). Thus, we have the following

Theorem. *Let* u_ε *be the solution to problem (5) and* $\{w_i\}$ *be the solution to discrete problem (6) on mesh* I_h *with* $n \geq n_1$, *where* $n_1 \in \mathbb{N}$ *is great enough and independent of* ε.

Then we have the second order convergence uniform in ε:

$$|u_\varepsilon(x_i) - w_i| \leq Mh^2, \qquad i=0,1,\dots,n.$$

4. PROOF OF THE UNIFORM CONSISTENCY

Let, besides (7), n satisfy $n > 6/\tau$. The we can prove:

$$|r_C u_\varepsilon(x_i)| \leq Mh^2, \qquad i=1,2,\dots,n-2,$$

$$|r_M u_\varepsilon(x_i)| \leq Mh^2, \qquad i=2,3,\dots,n-2, \tag{8}$$

(note that at x_1, L_C^h is applied),

$$|r_u u_\varepsilon(x_{n-1})| \le Mh,$$

where $r_C u_\varepsilon(x_i) = (Lu_\varepsilon)(x_i) - L_C^h u_\varepsilon(x_i)$, etc. Inequality (8) will be proved only, since the other ones can be proved analogously. For the technique cf. [2], [3], [7].

Let $i=2,3,\dots,n-2$. We have to prove that

(9a) $$R'' := \varepsilon^2 |u_\varepsilon''(x_{i+1/2}) - D_M'' u_\varepsilon(x_{i+1/2})| \le Mh^2,$$

(9b) $$R' := |(xu_\varepsilon)'(x_{i+1/2}) - D_M' x_{i+1/2} u_\varepsilon(x_{i+1/2})| \le Mh^2,$$

(9c) $$R^o := |u_\varepsilon(x_{i+1/2}) - D_M^o u_\varepsilon(x_{i+1/2})| \le Mh^2.$$

The following estimates are valid:

(10a) $$R'' \le Mh^2\varepsilon^2 (\lambda'(t_{i+2})^3/\lambda'(t_{i-1})) U_{i-1,i+2}^{(4)},$$

(10b) $$R' \le Mh^2 \lambda'(t_{i+1})^2 \max_{x_i \le x \le x_{i+1}} |(xu_\varepsilon(x))'''|,$$

(10c) $$R^o \le M((h_{i+1}-h_i)U_{i-1,i+1}^{(1)} + h^2\lambda'(t_{i+1})^2 U_{i-1,i+1}^{(2)}),$$

where

$$U_{k,\ell}^{(s)} = \max_{x_k \le x \le x_\ell} |u_\varepsilon^{(s)}(x)|, \qquad s=1,2,4.$$

The other estimates which we shall use are:

(11a) $$R'' \le M\varepsilon^2 U_{i-1,i+2}^{(2)},$$

(11b) $$R' \le M(1/h_{i+1}) \left| \int_{x_i}^{x_{i+1}} (s-x_{i+1/2})(su_\varepsilon(s))''ds \right|,$$

$$\text{(11c)} \qquad R^O \le M \int_{x_{i-1}}^{x_{i+1}} |u_\varepsilon'(x)|ds.$$

Let $j \in N$ be given by $t_{j-1} < \tau/2 \le t_j$. Because of $n > 6/\tau$, we have $j \ge 4$ and $t_{j+2} < \tau/2+3h < \tau$. Then

$$x_{j+2} = \psi(t_{j+2})^{2/p} \le m_o\varepsilon.$$

The proof will be given in the following steps:

1^O $\quad i = 2(1)j$

2^O $\quad i = j+1(1)n-2$

2^O1 $\quad t_{i-1} \ge \tau+h$

2^O2 $\quad \tau+h > t_{i-1} \ge \tau$

$2^O2a)$ $\quad h \le \varepsilon^{p/4}$

$2^O2b)$ $\quad h \ge \varepsilon^{p/4}$

2^O3 $\quad t_{i-1} < \tau,\ t_{i-1} \le q-4h$

2^O4 $\quad q-4h < t_{i-1} < \tau.$

The estimates from Lemma 1 will be used. Since the exponential terms can be treated as in [2] we shall here consider non-exponential terms only.

First we shall need

Lemma 2. *In cases 1^O, 2^O1, $2^O2a)$ and 2^O3, we have*

$$\text{(12)} \qquad \lambda^{(k)}(t_{i+2})/\lambda^{(k)}(t_{i-1}) \le M, \quad k=0,1.$$

Proof. Let us prove (12) for k=0. In case $2^{O}1$, we have to prove

$$Q := \pi(t_{i+2})/\pi(t_{i-1}) \leq M.$$

Let $s = t_{i-1} - \tau \geq h$. Then $Q \leq Mg(s)$ with

$$g(s) = (s+2h+\varepsilon^{p/4})/(s+\varepsilon^{p/4}).$$

Since $g'(s)<0$, it follows that

$$g(s) \leq g(h) \leq M,$$

and this part of proof is completed.

In case $2^{O}2a)$, we have

$$Q \leq M(h+\varepsilon^{p/4})/\varepsilon^{p/4} \leq M$$

and (12) holds again.

In case $2^{O}3$, we have to prove

$$\psi(t_{i+2})/\psi(t_{i-1}) \leq M, \tag{13}$$

since $t_{i+2}<q$ and $\lambda_1(t_{i+2}) \leq \psi(t_{i+2})$. Furthermore, in this case we have

$$q-t_{i+2} \geq (q-t_{i-1})/4,$$

and (13) follows immediately.

In case 1^{O}, we have to prove (13) as well. Now

$$(q-t_{i-1})/(q-t_{i+1}) \leq 1 + 3h/(\tau/2-3h) \leq M,$$

and we get (13) again.

Inequality (12) for k=1 can be proved in the same way since

$$\lambda'(t) = (p/2)\lambda_1(t)^{p/2-1}\lambda_1'(t). \quad \square$$

Now we continue with proving (9).

1° We use Lemma 2 for k=1, estimates (2a, 3a) and facts that

$$\lambda'(t_{i+2}) \le M\varepsilon,$$

$$h_{i+1}-h_i \le Mh^2\lambda''(t_{i+1}) \le Mh^2\varepsilon$$

to get (9) from (10).

2° By considering the non-exponential terms in (2b, 3b), we conclude that when using (10) it is sufficient to prove

$$P'' := (\lambda'(t_{i+2})^3/\lambda'(t_{i-1}))\lambda(t_{i-1})^{p/2} \le M,$$

$$P' := \lambda'(t_{i+1})^2\lambda(t_{i-1})^{p-2} \le M,$$

(14) $$P^{o} := (h_{i+1}-h_i)\lambda(t_{i-1})^{p-1} \le Mh^2,$$

($P^{(k)}$ is derived from the estimates (10) of $R^{(k)}$, k=0,1,2). In cases 2°1, 2°2a) and 2°3, we get

$$P^{(k)} \le M, \quad k=1,2,$$

because of Lemma 2 and (4b). In cases 2°1 and 2°2a), we have

(15) $$h_{i+1}-h_i \le h^2\lambda''(t_{i+1})$$

and because of (4c) and Lemma 2 we get (14). Now let us prove (14) in case 2°3. If $t_i \le \tau$, then because of $\pi(t_{i+1}) \le \psi(t_{i+1})$ (15) holds again with $\lambda=\psi^{2/p}$. Because of

(16) $q-t_{i+1} \geq (q-t_{i-1})/4,$

it follows that

$$p^o \leq Mh^2\varepsilon^p/(q-t_{i-1})^4,$$

but since

$$q-t_{i-1} > q - \tau = M\varepsilon^{p/4}$$

we obtain (14). If $t_i>\tau>t_{i-1}$ we use

$$h_{i+1}-h_i = \pi(t_{i+1})^{2/p}-2\pi(t_i)^{2/p}+\psi(t_{i-1})^{2/p} \leq$$

$$\leq \psi(t_{i+1})^{2/p}-2\psi(t_i)^{2/p}+\psi(t_{i-1})^{2/p} +$$

$$+ 2(\psi(t_i)^{2/p}-\pi(t_i)^{2/p}) \leq h^2(\psi^{2/p})''(t_{i+1}) +$$

$$+ (4/p)\psi(t_i)^{2/p-1}(\psi(t_i)-\pi(t_i)) \leq$$

$$\leq Mh^2((\psi^{2/p})''(t_{i+1})+\psi(t_i)^{2/p-1}\psi''(t_i)).$$

Now from (16) it follows that

$$p^o \leq Mh^2\varepsilon^p/(q-t_{i-1})^4 \leq Mh^2 .$$

There remains to prove (9) in cases $2^o2b)$ and 2^o4. Now we shall use estimates (11). Again we shall consider the nonexponential terms from (2b, 3b) only. Thus, it is sufficient to prove:

$$S'': = \varepsilon^2\lambda(t_{i-1})^{p/2} \leq Mh^2,$$

$$S': = \int_{x_i}^{x_{i+1}} s^{p-1}ds \leq Mh^2,$$

$$S^{o}: = \int_{x_{i-1}}^{x_{i+1}} s^{p-1}ds \le Mh^2,$$

($S^{(k)}$ is derived from the estimates (11) of $R^{(k)}$, k=0,1,2).

$2^{o}2b)$ The proof for S^{o} and S' follows immediately, since

$$S^{(k)} \le x_{i+1}^{p} = \pi(t_{i+1})^2 \le M(h+\varepsilon^{p/4})^2 \le Mh^2, \quad k=0,1.$$

For S'' we have

$$S'' \le M\varepsilon^2\pi(\tau+h)^{(2/p)(p-2)} \le M\varepsilon^{1+p/2} \le Mh^2.$$

$2^{o}4$ Now because of $q-4h<t_{i-1}<\tau=q-M\varepsilon^{p/4}$ we have $\varepsilon^{p/4} \le Mh$. Since

$$S'' \le \varepsilon^2(\psi(\tau))^{(2/p)(p-2)} \le M\varepsilon^{p} \le Mh^2$$

$$S^{(k)} \le x_{i+1}^{p} \le M(\psi(\tau)+h)^2 \le Mh^2, \quad k=0,1,$$

(8) is proved. □

5. NUMERICAL RESULTS

Now we shall give some numerical results for the test problem:

$$-\varepsilon^2u''-xu'+2u=f(x),$$

$$u(0)=1 \;, u(1)=1+\exp(-1/\varepsilon^2),$$

with the exact solution $u=x+\exp(-(x/\varepsilon)^2)$.

The maximum point-wise error is denoted by E. By P we denote the percentage of mesh steps in interval $[0,\varepsilon]$. We take that n=50.

In Table 1 we give the results of our method. We take that p=1, A=0.5, q=0.5 and obtain P=32%.

Table 1

ε	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}
$10^3 E$	2.77	4.49	4.99	5.24	5.30

We compare these results with the results obtained by the up--wind scheme on the mesh from [3] with p=1: P=30% and E=0.018 for all ε from Table 1.

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REZIME

UNIFORMNI NUMERIČKI METOD DRUGOG REDA
ZA PROBLEM SA POVRATNOM TAČKOM

Posmatra se singularno perturbovani konturni problem sa povratnom tačkom. Neekvidistantna generalizacija šeme Guščina-Ščenikova koristi se na specijalnoj mreži diskretizacije i dokazuje se drugi red konvergencije, uniformne po malom perturbacionom parametru.

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