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ON SOME CLASSES OF PERMUTABLE n-GROUPOIDS AND n-QUASIGROUPS

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Abstract. Some classes of H-permutable and exactly H-permutable n-groupoids and n-quasigroups are considered. If (Q,f) is an n-groupoid and H a subgroup of the symmetric group S_{n+1}, such that for every $\sigma \in H$ $f(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1,\dots,x_n) = x_{n+1}$ for all $x_1,\dots,x_{n+1} \in Q$, then (Q,f) is called H-permutable. Moreover, if H consists of all permutations with the given property, then (Q,f) is exactly H-permutable. It is proved that every H-permutable n-groupoid is an n-quasigroup iff H is a transitive permutation group. For some groups H a class of exactly H-permutable n-quasigroups of any prime order $p>n+1$ is constructed, which establishes a conjecture of Hoffman [2] for such groups H. A corollary of this is the existence of n-quasigroups of every order $>n+1$ with exactly $\frac{(n+1)!}{n_1!\dots n_k!}$ conjugacy classes, where $n_1,\dots,n_k$ are arbitrary positive integers such that $n_1+\dots+n_k \le n+1$. The existence of exactly cyclic n-quasigroups of prime order $p \equiv 1 \pmod{n+1}$ and infinite order is also proved.

1. Notations and Definitions

We shall give some basic definitions and notations.

The sequence $x_m, x_{m+1},\dots,x_n$ we shall denote by x_m^n. If $m>n$, then x_m^n will be considered empty.

An n-ary groupoid (n-groupoid) (Q,f) is called an n-quasigroup iff the equation $f(a_1^{i-1},x,a_{i+1}^n) = b$ has a unique solution for every $a_1^n, b \in Q$ and every $i \in N_n = \{1,\dots,n\}$. The cardinality of the set Q will be denoted by $|Q|$. An n-groupoid (Q,f) is called idempotent iff $f(x,\dots,x) = x$ for all $x \in Q$.

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By S_n we denote the symmetric group of degree n.

If $\sigma \in S_n$, then $x_{\sigma i} x_{\sigma(i+1)}, \dots, x_{\sigma j}$ will be denoted by $x_{\sigma i}^{\sigma j}$. If i>j, then $x_{\sigma i}^{\sigma j}$ is considered empty.

Let $\sigma \in S_{n+1}$. An n-groupoid (Q,f) is called σ-permutable ([1]) iff for all $x_1^n \in Q$

$$f(x_1^n) = x_{n+1} \Leftrightarrow f(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)}.$$

If (Q,f) is an n-groupoid, then the set H of all permutations $\sigma \in S_{n+1}$ such that (Q,f) is σ-permutable is a subgroup of S_{n+1}. The set H of all such permutations will be denoted by $\Pi(f)$.

Let H be a subgroup of S_{n+1}. If an n-groupoid (Q,f) is σ-permutable for every $\sigma \in H$, then (Q,f) is called a H-permutable n-groupoid. Moreover, if $\Pi(f) = H$, then (Q,f) is called exactly H-permutable n-groupoid.

H-permutable n-groupoid for $H = S_{n+1}$ is called totally symmetric (TS) and for $H = C_{n+1}$, where C_{n+1} is the cyclic group generated by the cycle (12...n+1), it is called a cyclic n-quasigroup ([5]). For each subgroup H of S_{n+1}, we define $\Lambda(H)$, the spectrum of H, to be the set of all positive integers q for which there exists an n-quasigroup (Q,f) of order q with $\Pi(f) = H$.

If (Q,f) is an n-quasigroup, $\sigma \in S_{n+1}$, then the n-quasigroup f^{σ} defined by

$$f^{\sigma}(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1}$$

is called a σ-conjugate (or parastrophe) of f. The set $\{(Q,f^{\sigma}) \mid \sigma \in S_{n+1}\} = C(Q,f)$ is called the set of conjugacy classes of (Q,f) ([3],[4]). $|C(Q,f)|$ is called the conjugacy class number of (Q,f). If (Q,f) is exactly H-permutable n-quasigroup, then $|C(Q,f)| = |S_{n+1}|:|H|$.

2. H-permutable n-groupoids and n-quasigroups

Theorem 1. Let H be a nontrivial subgroup of the symmetric group S_{n+1}. Every H-permutable n-groupoid is an n-quasigroup iff H is a transitive permutation group.

P r o o f. If H is a transitive permutation group, then Corollary 2 from [1] implies that every H-permutable n-groupoid is an n-quasigroup.

Now we assume that H is not a transitive permutation group. If for every $k \in N_{n+1}$, there exists $\sigma \in H$ such that $\sigma k = n+1$, then it can easily be proved that H must be transitive. Hence there exists $k \in N_{n+1}$ such that there is no permutation in H which mapps k to n+1.

Let $P = \{\sigma(k) \mid \sigma \in H\}$ and $R = N_{n+1} \setminus P$. Let (Q,+) be a commutative group, $|Q| \neq 1$. If we denote $P = \{a_1, \dots, a_i\}$, $R = \{b_1, \dots, b_m\}$, $b_m = n+1$, and if

$a \in Q$ is an arbitrary element, then we define an n-groupoid (Q,f) by

$$f(x_1^n) = -x_{b_1} - \ldots - x_{b_{m-1}} + a.$$

We shall prove that f is an H-permutable n-groupoid which is not an n-quasigroup. Since $|P| \geq 1$, f is not an n-quasigroup. If for some $\sigma \in H$ and some $a_j \in P$, $\sigma(a_j) = b_s$, since there exists $\tau \in H$ such that $\tau(k) = a_j$, it follows $\sigma\tau(k) = b_s$, which is a contradiction. Hence for every $\sigma \in H$ and every $x \in P$, $\sigma(x) \in P$. Also, for every $\sigma \in H$ and every $y \in R$, $\sigma(y) \in R$ (since $\sigma(b_p) = a_q$ implies $\sigma^{-1}(a_q) = b_p$, and that case was considered earlier). So, (Q,f) is an H-permutable n-groupoid.

REMARK 1. For a nontransitive subgroup H of S_{n+1} the H-permutable n-groupoid (Q,f) constructed in the proof of the preceding theorem need not be exactly H-permutable, but f is never TS, since P and R are always nonempty sets. In fact, for every $\sigma \in S_{n+1}$ such that $\sigma(P)=P$, f is σ-permutable and for every $\sigma \in S_{n+1}$ which mapps an element from P into R, it can be proved that f is not σ-permutable. Hence if $K \subseteq S_{n+1}$ is the set of all permutations which map P onto P, (Q,f) is exactly K-permutable.

In [2] D.G. Hoffman has proved that there exist exactly H-permutable n-quasigroups of order mp, where m,p are arbitrary positive integers such that $m>n$ and $p \geq 2$. In particular, such n-quasigroups exist for every composite order $q \geq (n+1)^2$. These results also imply the existence of n-quasigroups of the given orders with exactly $(n+1)!/|H|$ conjugacy classes for every subgroup $H \subseteq S_{n+1}$.

In [2] also the following conjecture was made:

For each subgroup H of S_{n+1}, $\Lambda(H)$ consists of all but finitely many positive integers.

Now we shall give some constructions of exactly H-permutable n-quasigroups. In view of the results from [2], we shall restrict ourselves to n-quasigroups of prime orders, although the obtained results can be easily extended to some composite orders.

Theorem 2. Let H be a subgroup of S_{n+1}. If there exist disjoint sets $R_1,\ldots,R_k \subseteq N_{n+1}$ such that for every $\sigma \in H$, $\sigma(R_i) = R_i$, $i=1,\ldots,k$, for every $x \in N_{n+1} \setminus (R_1 \cup \ldots \cup R_k)$, $\sigma(x) = x$, and H contains all permutations from S_{n+1} with the given properties, then there exists an exactly H-permutable n-quasigroup of order p, where $p>n+1$ is any prime.

P r o o f. Let p be a prime and let $GF(p) = Q$ be the Galois field of order p. We define an n-groupoid (Q,f) by

$$f(x_1^n)=x_{n+1} \Leftrightarrow \alpha_1(x_{r_{11}}+\ldots+x_{r_{1j_1}})+\ldots+\alpha_k(x_{r_{k1}}+\ldots+x_{r_{kj_k}})+\beta_1 x_{s_1}+\ldots+\beta_m x_{s_m}$$

where $R_i=\{r_{i1},\ldots,r_{ij_i}\}$, $i=1,\ldots,k$, $N_{n+1}\setminus(R_1\cup\ldots\cup R_k)=\{s_1,\ldots,s_m\}$ and $\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_m,\gamma$ are elements of Q which are mutually different and different from 0 and 1.

It is obvious that (Q,f) is an H-permutable n-quasigroup and it remains to prove that $\Pi(f)=H$. Let $\sigma\notin H$. We shall consider two cases.

a) If there exist $a\in R_i$, $b\in R_j$, $i\neq j$, such that $\sigma(a)=b$, then for $x_a=\gamma/\alpha_i$ and $x_t=0$ for all $t\neq a$, it follows $f(x_1^n)=x_{n+1}$, but $f(x_{\sigma^{-1}1}^{\sigma^{-1}n})\neq x_{\sigma^{-1}(n+1)}$, which means that f is not σ^{-1}-permutable, hence f is not σ-permutable.

b) If $\sigma(R_i)=R_i$, $i=1,\ldots,n$, and there exists $s_t\in N_{n+1}\setminus(R_1\cup\ldots\cup R_k)$ such that $\sigma(s_t)=s_u$, $t\neq u$, then for $x_{s_u}=\gamma/\beta_u$ and $x_v=0$ for all $v\neq s_t$, it follows $f(x_1^n)=x_{n+1}$, but $f(x_{\sigma 1}^{\sigma n})\neq x_{\sigma(n+1)}$. So, f is not σ-permutable, which completes the proof.

REMARK 2. The preceding theorem establishes the conjecture from [2] for all subgroups H of S_{n+1} of the form described in the theorem.

In Theorem 2 a construction of an exactly H-permutable n-quasigroup of any prime order $p>n+1$ was given, where $|H|=|R_1|!\ldots|R_k|!$ and $|R_1|+\ldots+|R_k|\leq n+1$ and H satisfies the conditions from Theorem 2. Such subgroup H exists for any choice of disjoint sets $R_1,\ldots,R_k\subseteq N_{n+1}$, hence we get the following corollary which extends results from [3].

Corollary 1. There exists an n-quasigroup of prime order $p>n+1$ with exactly $\frac{(n+1)!}{n_1!\ldots n_k!}$ conjugacy classes, where $n_1,\ldots,n_k$ are arbitrary positive integers such that $n_1+\ldots+n_k\leq n+1$.

3. Cyclic n-quasigroups

Now we shall investigate the spectrum of cyclic n-quasigroups (Q,f) ([5]) with the property that $\Pi(f)=C_{n+1}$, where C_{n+1} is a subgroup of S_{n+1} generated by the cycle $(12\ldots n+1)$. Such cyclic n-quasigroups we shall call exactly cyclic n-quasigroups.

We shall consider only exactly cyclic n-quasigroups of prime order, since the existence of such n-quasigroups of composite orders follows from [2].

Theorem 3. If p is a prime such that $p \equiv 1 \pmod{n+1}$, then there exists an exactly cyclic n-quasigroup of order p.

P r o o f. Let $GF(p)=Q$ be the Galois field of order p and $p \equiv 1 \pmod{n+1}$. Since $p-1=k(n+1)$ and the multiplicative group of the field is cyclic of order p-1, there exists an element $a \in Q$ such that $a^{n+1}=1$ and $a^s \neq 1$ for all $s < n+1$. We define an n-operation f on Q by

$$f(x_1^n) = -ax_1-a^2x_2-\ldots-a^nx_n,$$

and we shall prove that (Q,f) is an exactly cyclic n-quasigroup.

It is not difficult to see that f is a cyclic n-quasigroup.

Let $\tau \in S_{n+1} \backslash C_{n+1}$. Since $C_{n+1}=\{\sigma_0,\ldots,\sigma_n\}$, where σ_k: $i \to i+k$, $i=1,\ldots,n+1$, $k=0,\ldots,n$, and $i+k$ is taken modulo n+1, it follows that there exist elements $i,j,k,m \in N_{n+1}$ such that $i-j \not\equiv k-m \pmod{n+1}$ and $\tau(j) = i$, $\tau(m) = k$. If we put $b_i = 1$, $b_k = -a^{i-k}$, $b_t = 0$ for all other values of $t \in N_{n+1}$, then $f(b_1^n) = b_{n+1}$.

If we assume that f is τ-permutable, then $f(b_{\tau 1}^{\tau n}) = b_{\tau(n+1)}$, which implies $-a^j+a^m a^{i-k} = 0$ and $a^{i-j+m-k}=1$. The last equality gives $i-j \equiv m-k \pmod{n+1}$, which is a contradiction. Hence f is not τ-permutable, that is, f is an exactly cyclic n-quasigroup.

REMARK 3. Since $|C_{n+1}| = n+1$, Theorem 3 proves the existence of an n-quasigroup of order p with exactly n! conjugacy classes, where p is a prime such that $p \equiv 1 \pmod{n+1}$. This extends results from [3].

REMARK 4. It can be seen easily that the exactly cyclic n-quasigroup constructed in the proof of Theorem 3 is idempotent.

In combinatorial investigations finite structures are of primary importance, but from the algebraic point of view it is interesting to discuss also the existence of infinite structures.

Theorem 4. There exist infinite exactly cyclic n-quasigroups.

P r o o f. Let $F^{n,n}$ be the set of all matrices of order n over a field F. If A is the companion matrix of the polynomial $\lambda^n+\lambda^{n-1}+\ldots+\lambda+1$, then $A^{n+1}=E$ (the identity matrix) and $A^k \neq E$, $k=1,\ldots,n-1$. We define an n-groupoid $(F^{n,n},f)$ by

$$f(X_1^n) = -AX_1-A^2X_2-\ldots-A^nX_n.$$

A is regular, hence f is an n-quasigroup. It is easy to see that f is cyclic and moreover, as in Theorem 3, it can be proved that f is exactly cyclic n-quasigroup.

In the preceding theorem if F is a finite field of order p^m, then an exactly cyclic n-quasigroup f of order p^{mn^2} is obtained, if F is the field of rationals or reals we obtain a countable or uncountable infinite exactly cyclic n-quasigroup f.

REMARK 5. The exactly cyclic n-quasigroup defined in the proof of Theorem 4 is idempotent.

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REZIME

Razmotrene su neke klase H-permutabilnih i tačno H-permutabilnih n-grupoida i n-kvazigrupa. Ako je (Q,f) n-grupoid, a H podgrupa simetrične grupe S_{n+1}, takva da je za svako $\sigma \in H$ $f(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1,\dots,x_n) = x_{n+1}$ za sve $x_1,\dots,x_{n+1} \in Q$, onda se (Q,f) naziva H-permutabilnim. Ako se H sastoji od svih permutacija sa datom osobinom, onda je (Q,f) tačno H-permutabilan. Dokazano je da je svaki H-permutabilni n-grupoid n-kvazigrupa ako i samo ako je H tranzitivna grupa permutacija. Za neke grupe H konstruisana je klasa tačno H-permutabilnih n-kvazigrupa bilo kog prostog reda $p > n+1$, što potvrđue hipotezu Hoffmana ([2]) za takve grupe H. Posledica prethodnog je egzistencija n-kvazigrupa svakog reda $>n+1$ sa tačno $(n+1)!/n_1!\dots n_k!$ klasa konjugovanosti, gde su $n_1,\dots,n_k$ proizvoljni prirodni brojevi takvi da je $n_1+\dots+n_k \le n+1$. Dokazana je takođe i egzistencija tačno cikličkih n-kvazigrupa prostog reda $p \equiv 1(\mathrm{mod}(n+1))$ i beskonačnog reda.

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