Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 18,1,129-142(1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

COMMON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS

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ABSTRACT

In this paper, we give some common fixed point theorems for weakly commuting self-mappings with a family of self--mappings on 2-metric spaces. Our main theorems generalize, improve and unify some of the results of Khan-Fisher, Kubiak, Naidu-Prasad, Rhoades, Singh-Tiwari-Gupta and many others.

1. INTRODUCTION AND PRELIMINARIES

The concept of 2-metric spaces has been investigated initially by Gähler in a series of papers [1]-[3] and has been developed extensively by Gähler and many others.

AMS Mathematics Subject Classification (1980): 54H25.
Key words and phrases: Common fixed point theorems, weakly commuting mapping.

A 2-metric space is a set X with a real-valued function d on X x X x X satisfying the following conditions:

(M_1) For two distinct points x, y in X, there exists a point z in X such that $d(x,y,z)\neq 0$,

(M_2) $d(x,y,z)=0$ if at least two of x, y, z are equal,

(M_3) $d(x,y,z)=d(x,z,y)=d(y,z,x)$,

(M_4) $d(x,y,z)\leq d(x,y,u)+d(x,u,z)+d(u,y,z)$, for all x, y, z, u in X.

Then d is called a 2-metric for the space X and (X,d) is called a 2-metric space. It has been shown by Gähler [1] that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric d which is continuous in all of its arguments will be called continuous.

On the other hand, a number of mathematicians ([4]-[15], [17]-[28]) have studied the aspects of fixed point theory in the setting of the 2-metric spaces. They have been motivated by various concepts already known for ordinary metric spaces and have thus introduced analogues of various concepts in the framework of the 2-metric spaces. Especially, Khan [7] and Naidu-Prasad [17] introduced the concept of weakly commuting pairs of self-mappings on a 2-metric space and the notion of weak continuity of a 2-metric, respectively, and they have proved several common fixed point theorems by using the weakly commuting pairs of self-mappings on a 2-metric space and the weak continuity of a 2-metric. In this paper, we shall derive some common fixed point theorems for weakly commuting self--mappings with a family of self-mappings on 2-metric spaces. Our main results generalize, improve and unify some of results of Khan-Fisher [10], Kubiak [13], Naidu-Prasad [17], Rhoades [19], Sinhg-Tiwari-Gupta [26], Singh-Bam [27] and many others.

Now we shall give some definitions:

DEFINITION 1.1. *A sequence* $\{x_n\}$ *in a 2-metric space* (X,d) *is said to be convergent to a point* x *in* X *if* $\lim_{n\to\infty} d(x_n,x,z)=0$ *for all* z *in* X. *Then* x *is called the limit of the sequence* $\{x_n\}$ *in* X.

DEFINITION 1.2. *A sequence* $\{x_n\}$ *in a 2-metric space* (X,d) *is said to be a Cauchy sequence if* $\lim_{m,n\to\infty} d(x_m,x_n,z)=0$ *for all* z *in* X.*

DEFINITION 1.3. *A 2-metric space* (X,d) *is said to be complete if every Cauchy sequence in* X *is convergent.*

DEFINITION 1.4. *A 2-metric* d *on a set* X *is said to be weakly continuous on* X *if every convergent sequence in* X *is a Cauchy sequence.*

It is known that a 2-metric d on a set X is weakly continuous on X if and only if for every convergent sequence $\{x_n\}$ in X, the sequence $\{d(x_n,x_{n+1},z)\}$ converges to 0 for all z in X ([17]).

DEFINITION 1.5. *Let* S *and* T *be two mappings from a 2-metric space* (X,d) *into itself. Then a pair* (S,T) *is said to be weakly commuting on* X *if* $d(STx,TSx,z)\le d(Tx,Sx,z)$ *for all* z *in* X.

Note that a commuting pair (S,T) on a 2-metric space (X,d) is weakly commuting, but the converse is not true ([17]).

Throughout this paper, let (X,d) be a 2-metric space with a continuous 2-metric d. Let $\mathbb{N}$ and $\mathbb{R}^+$ be the sets of all natural numbers and non-negative real numbers, respectively, and F the family of mappings ϕ from $\mathbb{R}^+$ into itself such that each ϕ is non-decreasing, upper-semi-continuous and $\phi(t+)<t$ for every $t\in(0,\infty)$.

Let S and T be two mappings from a 2-metric space (X,d) into itself and $\{f_i\}$ a family of mappings f_i, $i\in\mathbb{N}$, from X into itself such that

* Note that in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X([17]).

(1) $f_i(X) \subseteq S(X) \cap T(X)$ for each $i \in \mathbb{N}$,

(2) $d(f_ix, f_jy, z) \leq \phi(\max\{d(Sx,Ty,z),\ d(Sx,f_ix,z),$
$d(Ty,f_jy,z),\ d(Ty,f_ix,z),\ d(Sx,f_jy,z\}\})$ for all x, y, z in X and $i, j \in \mathbb{N}$, $i \neq j$, where $\phi \in F$.

Then, by (1), for any arbitrary point x_o in X, we take a point x_1 in X such that $Tx_1 = f_1x_o$ and for this point x_1 there exists a point x_2 in X such that $Sx_2 = f_2x_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

(3) $$\begin{cases} y_{2n+1} Tx_{2n+1} = f_{2n+1}x_{2n} & \text{for every } n \in \mathbb{N}_o \text{ and} \\ y_{2n} = Sx_{2n} = f_{2n}x_{2n-1} & \text{for every } n \in \mathbb{N}, \end{cases}$$

where $\mathbb{N}_o = \mathbb{N} \cup \{0\}$.

By analogy with the notion of the asymptotic regularity of self-mappings on a metric space ([29]), we shall introduce the concept of the asymptotic regularity of self-mappings on a 2-metric space.

DEFINITION 1.6. *Let* S *and* T *be two mappings from a 2-metric space* (X,d) *into itself. Then a pair* (S,T) *is said to be asymptotically regular at* x_o *in* X *if* $\lim_{n\to\infty} d(y_n, y_{n+1}, z) = 0$ *for all* z *in* X, *where* $\{y_n\}$ *is the sequence in* X *defined by (3).*

Before proving and stating our main theorems, we need the following lemmas:

LEMMA 1.1.[16] *Let* $\phi \in F$ *and* $t_o > 0$. *If* $t_{n+1} \leq \phi(t_n)$ *for every* $n \in \mathbb{N}$, *then the sequence* $\{t_n\}$ *converges to* 0.

LEMMA 1.2. *For every* $i, j, k \in \mathbb{N}_o$, $d(y_i, y_j, y_k) = 0$, *where* $\{y_n\}$ *is the sequence in* X *defined by (3).*

PROOF. From (2), we have $d(y_{2n}, y_{2n+1}, y_{2n+2}) \leq$
$\leq \phi(d(y_{2n}, y_{2n+1}, y_{2n+2}))$ and $d(y_{2n+1}, y_{2n+2}, y_{2n+3}) \leq \phi(d(y_{2n+1}, y_{2n+2}, y_{2n+3}))$. Thus we have $d(y_n, y_{n+1}, y_{n+2}) \leq \phi(d(y_n, y_{n+1}, y_{n+2}))$ for every $n \in \mathbb{N}_o$. Since $\phi \in F$,

(4) $d(y_n, y_{n+1}, y_{n+2}) = 0$ for every $n \in \mathbb{N}_o$.

Now let $d_n(z)=d(y_n,y_{n+1},z)$ for all z in X and for every $n\in\mathbb{N}_o$. By (4) and (M_4) we deduce

(5) $d(y_n,y_{n+2},z)\leq d_n(z)+d_{n+1}(z)$ for all z in X and for every $n\in\mathbb{N}_o$.

From (2), (4) and (5), we obtain $d_{2n}(z)\leq\phi(d_{2n-1}(z)+d_{2n}(z))$ and $d_{2n+1}(z)\leq\phi(d_{2n}(z)+d_{2n+1}(z))$ for all z in X and for every $n\in\mathbb{N}_o$. Thus we have

(6) $d_n(z)\leq\phi(d_{n-1}(z)+d_n(z))$ for all z in X and for every $n\in\mathbb{N}_o$.

It follows from (6) that if $d_{n-1}(z)=0$, then $d_n(z)=0$. Since $d_o(y_o)=0$, by (6), $d_n(y_o)=0$ for every $n\in\mathbb{N}$. Since $d_{m-1}(y_m)=0$ for every $m\in\mathbb{N}$, by (6) again, $d_n(y_m)=0$ for every $n\geq m-1$. For $0\leq n<m-1$, since $d_{m-1}(y_{n+1})=d_{m-1}(y_n)=0$, we have

(7) $d_n(y_m)\leq d_n(y_{m-1})$.

Since $d_n(y_{n+1})=0$, from (7), we have $d_n(y_m)=0$ for every $0\leq n<m-1$. Thus we have $d_n(y_m)=0$ for every $m,\ n\in\mathbb{N}_o$. Since $d_{j-1}(y_i)=$ $=d_{j-1}(y_k)=0$ for every $i,\ j,\ k\in\mathbb{N}$ with $i<j$, $d(y_i,y_j,y_k)\leq$ $\leq d(y_i,y_{j-1},y_k)\leq\ldots\leq d(y_i,y_i,y_k)=0$. Therefore, we have $d(y_i,y_j,y_k)=0$ for every $i,\ j,\ k\in\mathbb{N}_o$.

LEMMA 1.3. *Let S and T be two mappings from a 2-metric space (X,d) into itself. If a pair (S,T) is asymptotically regular at x_o in X, then the sequence $\{y_n\}$ defined by (3) is a Cauchy sequence in X.*

PROOF. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. This means that there exist a point c in X and $\varepsilon>0$ such that for any $k\in\mathbb{N}$ there exist strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ with $k\leq n_k<m_k$ such that $d(y_{2n_k},y_{2m_k},c)\geq\varepsilon$ and $d(y_{2n_k},y_{2m_k-2},c)<\varepsilon$. Using Lemma 1.2. and (M_4) we have $d(y_{2n_k},y_{2m_k},c)-d(y_{2n_k},y_{2m_k-2},c)\leq d(y_{2m_k-2},y_{2m_k},c)\leq d_{2m_k-2}(c)+$ $+d_{2m_k-1}(c)$. Since $\{d(y_{2n_k},y_{2m},c)-\varepsilon\}$ and $\{\varepsilon-d(y_{2n_k},y_{2m_k-2},c)\}$

are sequences of nonnegative real numbers and the pair (S,T) is asymptotically regular at x_o in X, we have

(8) $$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}, c) = \varepsilon \text{ and } \lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-2}, c) = \varepsilon.$$

From (M_4), it is clear that

(9) $$|d(x,y,a)-d(x,y,b)| \le d(a,b,x)+d(a,b,y)$$

for all a, b, x, y in X. Taking $x=y_{2n_k}$, $y=c$, $a=y_{2m_k-1}$ and $b=y_{2m_k}$ in (9) and using the fact that the pair (S,T) is asymptotically regular at x_o in X, by Lemma 1.2 and (8), we obtain

(10) $$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}, c) = \varepsilon.$$

Since the pair (S,T) is asymptotically regular at x_o in X, from Lemma 1.2, (8), (9) and (10), we have

(11) $$\lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k}, c) = \varepsilon$$

and

(12) $$\lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k-1}, c) = \varepsilon.$$

By (2), we deduce $d(y_{2m_k}, y_{2n_k+1}, c) \le \phi(\max\{d(y_{2n_k}, y_{2m_k-1}, c),$ $d(y_{2n_k}, y_{2n_k+1}, c)$, $d(y_{2m_k-1}, y_{2m_k}, c)$, $d(y_{2n_k+1}, y_{2m_k-1}, c)$, $d(y_{2n_k}, y_{2m_k}, c)\})$. Taking limits on both sides as $k\to\infty$, by (8), (10), (11), (12) and the fact that the pair (S,T) is asymptotically regular at x_o in X and the 2-metric d is continuous, $\varepsilon \le \phi(\varepsilon)$, which contradicts $\phi(t+)<t$ for every $t \in (0,\infty)$. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X. By using Lemma 1.2 and (M_4), we have $d(y_{2n}, y_{2m+1}, z) \le d(y_{2n}, y_{2m}, z)+d_{2m}(z)$ and $d(y_{2n+1}, y_{2m+1}, z) \le d(y_{2n}, y_{2m+1}, z)+d_{2n}(z)$ for every $m, n \in \mathbb{N}_o$ and for all z in X. Therefore, $\{y_n\}$ is a Cauchy sequence in X.

II. MAIN THEOREMS

Now, we shall give our main theorems:

THEOREM 2.1. *Let* S *and* T *be two mappings from a complete 2-metric space* (X,d) *into itself such that either* S *or T is continuous. Then S and T have a common fixed point w if and only if there exist a family $\{f_i\}$ of mappings f_i, $i \in \mathbb{N}$, from X into itself and a mapping $\phi \in F$ such that (1), (2) and*

(13) *pairs (f_i, S) and (f_i, T) are weakly commuting for each $i \in \mathbb{N}$, and*

(14) *a pair (S,T) is asymptotically regular at x_0 in X.*

Further, w is a unique common fixed point of S,T and $\{f_i\}_{i \in \mathbb{N}}$.

PROOF. By Lemma 1.3, since (X,d) is a complete 2-metric space and $\{y_n\}$ is a Cauchy sequence in X, $\{y_{2n+1}\}$ is also a Cauchy sequence in X and converges to a point w in X. Since the pair (S,T) is asymptotically regular at x_0 in X, by (M_4), we have $d(y_{2n},w,z) \leq d(y_{2n},w,y_{2n+1}) + d(y_{2n},y_{2n+1},z) + d(y_{2n+1},w,z)$ for all z in X, which implies $\lim_{n\to\infty} d(y_{2n},w,z)=0$, that is, $\{y_{2n}\}$ converges also to w. Now, suppose that the self-mapping S on X is continuous. Then $\{Sy_{2n+1}\} = \{Sf_{2n+1}x_{2n}\}$ converges to Sw. Since (f_i, S) is a weakly commuting pair on X for each $i \in \mathbb{N}$, we have $d(f_{2n+1}Sx_{2n},Sw,z) \leq d(f_{2n+1}Sx_{2n},Sw,Sf_{2n+1}x_{2n}) + d(f_{2n+1}Sx_{2n},Sf_{2n+1}x_{2n},z) + d(Sf_{2n+1}x_{2n},Sw,z) \leq d(f_{2n+1}Sx_{2n},Sw,Sf_{2n+1}x_{2n}) + d(f_{2n+1}x_{2n},Sx_{2n},z) + d(Sf_{2n+1}x_{2n},Sw,z)$, which implies that $\{f_{2n+1}Sx_{2n}\}$ also converges to Sw as $n\to\infty$. By (2), we deduce $d(f_{2n+1}Sx_{2n},y_{2n},z) = d(f_{2n+1}Sx_{2n},f_{2n}x_{2n-1},z) \leq \leq \phi(\max\{d(SSx_{2n},Tx_{2n-1},z),\ d(SSx_{2n},f_{2n+1}Sx_{2n},z),\ d(Tx_{2n-1},f_{2n}x_{2n-1},z),\ d(Tx_{2n-1},f_{2n+1}Sx_{2n},z),\ d(SSx_{2n},f_{2n}x_{2n-1},z)\})$, which implies that, as $n\to\infty$, $d(Sw,w,z) \leq \phi(\max\{d(Sw,w,z),\ d(Sw,Sw,z),\ d(w,w,z),\ d(w,Sw,z),\ d(Sw,w,z)\}) = \phi(d(Sw,w,z))$ and so Sw=w. Using (2) again, we have $d(f_iw,f_{2n}x_{2n-1},z) \leq \leq \phi(\max\{d(Sw,Tx_{2n-1},z),\ d(Sw,f_iw,z),\ d(Tx_{2n-1},f_{2n}x_{2n-1},z),\ d(Sw,f_{2n}x_{2n-1},z),\ d(Tx_{2n-1},f_iw,z)\})$ for any odd $i \in \mathbb{N}$ and for all z in X and hence, by letting $n\to\infty$, we have $d(f_iw,w,z) \leq \leq \phi(\max\{d(Sw,w,z),\ d(Sw,f_iw,z),\ d(w,w,z),\ d(Sw,w,z),\ d(w,f_iw,z)\}) = \phi(\max\{0,\ d(w,f_iw,z),\ 0,\ 0,\ d(w,f_iw,z)\}) =$

$=\phi(d(w,f_iw,z))$, which yields $w=f_iw$ for any odd $i\in\mathbb{N}$.
By (1), we clearly have $w\in\bigcap_{n\in\mathbb{N}_0} f_{2n+1}(X)\subseteq T(X)$ and hence there exists a point w' in X such that $f_iw=Tw'=w$ for any odd $i\in\mathbb{N}$. Then, using (2), we deduce, for any even $j\in\mathbb{N}$ and for any odd $i\in\mathbb{N}$, $d(w,f_jw',z)=d(f_iw,f_jw',z)\leq\phi(\max\{d(Sw,Tw'z),$ $d(Sw,f_iw,z)$, $d(Tw',f_jw',z)$, $d(Sw,f_jw',z)$, $d(Tw',f_iw,z)\})=$ $=\phi(\max\{0,0,d(w,f_jw',z),\ d(w,f_jw',z),0\})=\phi(d(w,f_jw',z))$, which implies $f_jw'=w$ for any even $j\in\mathbb{N}$. Since (f_k,T) is a weakly commuting pair on X for any $k\in\mathbb{N}$, we have $d(Tf_jw',f_jTw',z)\leq$ $\leq d(Tw',f_jw',z)=d(w,w,z)=0$ for all z in X and hence

(15) $\qquad Tw = Tf_jw' = f_jTw' = f_jw.$

Moreover, (2) implies that for any odd $i\in\mathbb{N}$ and for any even $j\in\mathbb{N}$, $d(w,Tw,z)=d(f_iw,f_jw,z)\leq\phi(\max\{d(Sw,Tw,z),\ d(Sw,f_iw,z),$ $d(Tw,f_jw,z)$, $d(Sw,f_jw,z)$, $d(Tw,f_iw,z)\})=\phi(\max\{d(w,Tw,z),\ 0,$ $0,\ d(w,Tw,z),\ d(w,Tw,z)\}=\phi(d(w,Tw,z))$. Thus we have $w=Tw$ and so $f_jw=w$ for every even $j\in\mathbb{N}$. Therefore w is a common fixed point of S, T, and $\{f_i\}_{i\in\mathbb{N}}$.

Similarly, the proof can be completed in the case of the continuity of self-mapping T on X. Further, from (2), it is easy to see that w is the unique common fixed point of S, T and $\{f_i\}_{i\in\mathbb{N}}$.

Conversely let w be a common fixed point of S and T. Define $f_i(x)=w$ for any $i\in\mathbb{N}$ and for any x in X. Now, since $d(f_ix,f_jy,z)=d(w,w,z)=0$ for any $i,\ j\in\mathbb{N}$, $i\neq j$, and for any x, y, z in X, it is trivial that (2) holds for some $\phi\in F$ and, since $\{w\}=f_i(X)\subseteq S(X)\cap T(X)$ for any $i\in\mathbb{N}$, (1) holds also. It is easy to show that the pair (S,T) is asymptotically regular at x_0 in X. This completes the proof.

REMARK 1. If $\psi\colon(\mathbb{R}^+)^5\to\mathbb{R}^+$ is increasing in each coordinate variable and $\phi\colon\mathbb{R}^+\to\mathbb{R}^+$ is defined by

(16) $\qquad \phi(t) = \max\{\psi(t,t,t,at,bt);\ a+b=2\},$

then ϕ is an increasing mapping on $\mathbb{R}^+$ and

$$\psi(t_1,t_2,t_3,t_4,t_5) = \begin{cases} \phi(\max\{t_1,t_2,t_3,\frac{1}{2}(t_4+t_5)\}), \\ \text{if either } t_4=0 \text{ or } t_5=0, \\ \phi(\max\{t_1,t_2,t_3,t_4,t_5\}), \text{otherwise,} \end{cases}$$

for all t_1, t_2, t_3, t_4, $t_5 \in \mathbb{R}^+$. Conversely, if ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$ is an increasing mapping on $\mathbb{R}^+$ and ψ: $(\mathbb{R}^+)^5 \to \mathbb{R}^+$ is defined by

$$(17)\quad \psi(t_1,t_2,t_3,t_4,t_5) = \begin{cases} \phi(\max\{t_1,t_2,t_3,\frac{1}{2}(t_4+t_5)\}), \text{ if} \\ \text{either } t_4=0 \text{ or } t_5=0, \\ \phi(\max\{t_1,t_2,t_3,t_4,t_5\}), \text{otherwise,} \end{cases}$$

then ψ is increasing in each coordinate variable and (16) holds for all $t \in \mathbb{R}^+$ ([17]). Therefore, even if we replace (2) by the following inequality in Theorem 2.1;

$$(18)\qquad d(f_ix,f_jy,z) \le \psi(d(Sx,Ty,z),\ d(Sx,f_ix,z),\ d(Ty,f_jy,z),$$

$d(Ty,f_ix,z)$, $d(Sx,f_jy,z))$ for all x, y, z in X and i, $j \in \mathbb{N}$, $i \neq j$, where ψ: $(R^+)^5 \to R^5$ is increasing in each coordinate variable and $\phi(t+)<t$ for every $t \in (0,\infty)$ where ϕ is defined by (16), Theorem 2.1 is also true.

Consider the following inequality:

$$(19)\qquad d(f_ix,f_jy,z) \le \phi(\max\{d(Sx,Ty,z),\ d(Sx,f_ix,z),$$

$d(Ty,f_jy,z)$, $\frac{1}{2}\{d(Sx,f_jy,z)+d(Ty,f_ix,z)\}\})$ for all x, y, z in X and i, $j \in \mathbb{N}$, $i \neq j$, where $\phi \in F$.

REMARK 2. In Theorem 2.1, if we replace (2) by (19), then the condition of the asymptotic regularity of the pair (S,T) can be dropped. In fact, as in the proof of Lemma 1.2, it follows that $d(y_n,y_{n+1},y_{n+2})=0$ for every $n \in \mathbb{N}_o$. Hence, by (4) and (19), $d_{2n+1}(z) = d(y_{2n+1},y_{2n+2},z) = d(f_{2n+1}x_{2n},$ $f_{2n+2}x_{2n+1},z) \le \phi(\max\{d(Sx_{2n},Tx_{2n+1},z),\ d(Sx_{2n},f_{2n+1}x_{2n},z),$ $d(Tx_{2n+1},f_{2n+2}x_{2n+1},z),\ \frac{1}{2}\{d(Sx_{2n},f_{2n+2}x_{2n+1},z) + d(Tx_{2n+1},$ $f_{2n+1}x_{2n},z)\}\}) = \phi(\max\{d_{2n}(z),\ d_{2n}(z),\ d_{2n+1}(z),\ \frac{1}{2}\ d(y_{2n},y_{2n+2},$ $z)\}\}) \le \phi(\max\{d_{2n}(z),d_{2n}(z),\ d_{2n+1}(z),\ \frac{1}{2}\{d_{2n}(z)+d_{2n+1}(z)\}\})$

for all z in X. If $d_{2n+1}(z) > d_{2n}(z)$ for some $n \in \mathbb{N}_o$, since $d_{2n+1}(z) > 0$, $d_{2n+1}(z) \leq \phi(d_{2n+1}(z))$, which contradicts the factor of $\phi(t+) < t$ for any $t \in (0,\infty)$. Hence, $d_{2n+1}(z) \leq d_{2n}(z)$ and $d_{2n+1}(z) \leq \phi(d_{2n}(z))$ for every $n \in \mathbb{N}$. Similarly, we have $d_{2n+2}(z) \leq d_{2n+1}(z)$ and $d_{2n+2}(z) \leq \phi(d_{2n+1}(z))$ for every $n \in \mathbb{N}_o$. Therefore, we deduce that $\{d_n(z)\}$ is a non-decreasing sequence such that $d_{n+1}(z) \leq \phi(d_n(z))$ for every $n \in \mathbb{N}$. Therefore, by Lemma 1.1, $\lim_{n\to\infty} d_n(z) = 0$, that is, the pair (S,T) is asymptotically regular at x_o in X.

As an immediate consequence of Theorem 2.1, Remarks 1 and 2, we have the following:

COROLLARY 2.2.[19] *Let f be a mapping from a complete 2-metric space* (X,d) *into itself such that there exists a number* $h \in [0,1)$ *such that*

(20) $$d(fx,fy,z) \leq h \max\{d(x,y,z),\ d(x,fx,z),\ d(y,fy,z),\ d(x,fy,z),\ d(y,fx,z)\} \text{ for all } x, y, z \text{ in } X.$$

Then f *has a unique fixed point* w *in* X.

PROOF. From Theorem 2.1 and Remark 1, putting $f_i = f_j = f$, $S=T=i_X$: an identity mapping and defining $\psi: (\mathbb{R}^+)^5 \to \mathbb{R}^+$ by $\psi(t_1,t_2,t_3,t_4,t_5) = h\max\{t_1,t_2,t_3,t_4,t_5\}$, we have this corollary.

COROLLARY 2.3 [13] *Let* (X,d) *be a complete 2-metric space with* d *continuous and let* $S, T: X \to X$ *be a pair of continuous mappings. Then* S *and* T *have a common fixed point in* X *if and only if there exist mappings* $A, B: X \to S(X) \cap T(X)$ *such that* $AS = SA$, $BT = TB$, *satisfying*

(21) $$d(Ax,By,z) \leq h \max\{d(Sx,Ty,z),\ d(Sx,Ax,z),\ d(Ty,By,z)\ \tfrac{1}{2}\{d(Sx,By,a) + d(Ty,Ax,z)\}\}$$

for all x, y, z *in* X, *where* $0 < h < 1$. *Indeed,* A, B, S *and* T *have a unique common fixed point in* X.

REMARK 3. Theorem 1 of Singh-Tiwari-Gupta [26] is obtained as a special case of Corollary 2.3 and if, in Corollary 2.3, we take A=B and replace (21) by the following stronger condition: $d(Ax,Ay,z) \leq h \max\{d(Ax,Sx,z), d(Sx,Ty,z), d(Ay,Ty,z)\}$ for all x, y, z in X, where $0<h<1$, then we obtain Theorem 2 of Khan-Fisher [10].

COROLLARY 2.4 [17] *Let S and T be two mappings from a 2-metric space (X,d) into itself. If there exist two mappings f and g from X into itself, two sequences $\{x_n\}$, $\{y_n\}$ in X and a mapping $\phi \in F$ such that*

(22) $$d(fx,gy,z) \leq \phi(\max\{d(Sx,Ty,z), d(Sx,fx,z), d(Ty,gy,z), d(Ty,fx,z), d(Sx,gy,z)\})$$ *for all x, y, z in X,*

(23) $$fx_{2n} = Tx_{2n+1} = y_{2n} \quad and \quad gx_{2n+1} = Sx_{2n+1} = y_{2n+1}$$

for every $n \in \mathbb{N}_0$, *and*

(24) *the sequence $\{d(y_n,y_{n+1},z)\}$ converges to 0 for all z in X as $n \to \infty$, then $\{y_n\}$ is a Cauchy sequence. Suppose that the sequence $\{y_n\}$ converges to some point w in X. Then (a) if f, g, S and T are continuous at w and (f,S) and (g,T) are weakly commuting pairs on X, then w is a common fixed point of f, g, S and T, (b) if d is weakly continuous on X, S, T are continuous at w and (f,S), (g,T) are weakly commuting pairs on X, then w is a common fixed point of f, g, S and T.*

REMARK 4. Corollary 2.4 which is a part of Theorem 1.1 in [17] assumes the continuity of f, g, S and T, but our Theorem 2.1 is true even if either S or T is continuous and the weak continuity of 2-metric d is replaced by the asymptotically regular pair (S,T) of self-mappings S and T.

COROLLARY 2.5 *Let S and T be two mappings from a complete 2-metric space (X,d) into itself such that either S or T is continuous. Then S and T have a common fixed point w in X if and only if there exist two mappings f, g from X into itself and a mapping $\phi \in F$ such that (22),*

(25) $f(X) \cup g(X) \subseteq S(X) \cap T(X)$,

(26) *pairs* (f,S) *and* (g,T) *are weakly commuting on* X, *and*

(27) *a pair* (S,T) *is asymptotically regular at* x_o *in* X.

Further, w *is a unique common fixed point of* f, g, S *and* T.

As in the proof of Theorem 2.1, we can prove the following theorem:

THEOREM 2.6 *Let* f, g, S *and* T *be four mappings from a complete 2-metric space* (X,d) *into itself such that either* f *or* g *or* S *or* T *is continuous and a pair* (S,T) *is asymptotically regular at* x_o *in* X. *Then* f, g, S *and* T *have a common fixed point in* X *if and only if there exists a mapping* $\phi \in F$ *such that (22), (26) and*

(28) $f(X) \subseteq S(X)$ *and* $g(X) \subseteq T(X)$.

Further, w *is a unique common fixed point of* f, g, S *and* T.

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REZIME

ZAJEDNIČKE NEPOKRETNE TAČKE ZA SLABO KOMUTIRAJUĆA PRESLIKAVANJA

U ovom radu date su neke teoreme o zajedničkoj nepokretnoj tački za slabo komutirajuća preslikavanja nad 2-metričkim prostorima. Rezultati uopštavaju i unificiraju neke rezultate Khana i Fishera, Kubiaka, Naidua i Prasada, Rhoadesa, Singha, Tivaria i Gupte i drugih.

Received by the editors March 16, 1987.