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BIJECTIVE MAPS WHICH ARE CLOSE TO PARTIAL ISOMETRIES

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ABSTRACT

In this paper it is introduced the notion of partial isometry from a set X onto a set Y which are endowed with the families of pseudometrics. It is proved an estimation for the closeness of a partial linear isometry to a bijective map M from a locally convex space onto sequential complete Hausdorff locally convex space.

1. INTRODUCTION

The classical problem of Hyers and Ulam [2] asked whether any map M from one real Banach space X onto another real Banach space Y such that for some $\varepsilon > 0$

$$|\,\|M(x) - M(y)\| - \|x - y\|\,| < \varepsilon, \quad (x,y \in X),$$

is "close" to an isometry. This problem was recently solved by

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Gevirtz [1] in the sense that for surjective map M there exists a linear surjective isometry J such that

$$\|M(x) - J(x)\| \leq 5\varepsilon, \quad (x \in X).$$

He used some results of A. Vogt [7], who considered Mazur-Ulam theorem [4].

We shall use the approach of Lindenstrauss, Szankowski [3] to obtain estimation for closeness of an partial linear isometry of a bijective map M from a locally convex space X onto sequential complete Hausdorff locally convex space. The notion of partial isometry is introduced in this paper as a generalization of the isometry.

2. PARTIAL ISOMETRIES

First, we generalize the notion of isometry on metric spaces.

Definition 1. *Let* X *be a nonempty set endowed with a family of pseudometrics* $\{d_i\}_{i\in I}$ *(will be denoted by* $(X,\{d_i\}_{i\in I})$*) and let* Y *be another nonempty set endowed with a family of pseudometrics* $\{D_j\}_{j\in J'}$. *A map* J *from* X *onto* Y *is a* (I,J')*-isometry, or if there is no posibility for misunderstanding, only partial isometry, if for every* $j \in J'$ *there exists* $i \in I$ *such that*

$$D_j(J(x),J(y)) = d_i(x,y), \quad (x,y \in X).$$

Since each uniformity $\mathcal{U}$ for a set X is generated by the family $\mathcal{P}$ of all pseudo-metrics which are uniformly continuous on X × X ($\mathcal{P}$ is usually called the gage of $\mathcal{U}$) we can introduce by Definition 1. the notion of partial isometry between two uniform spaces.

We have the following obvious properties of partial isometries.

Proposition 1. *Let X,Y and Z be nonempty sets endowed with the families of pseudometrics* $\{d_i\}_{i\in I}$, $\{D_j\}_{j\in J'}$, $\{r_s\}_{s\in S}$, *respectively. If* J_1 *is an (I,J)-isometry from X onto Y and* J_2 *be a (J,S)-isometry from Y onto Z, then* $J_2 \circ J_1$ *is an (I,S)-isometry from X onto Z.*

Proposition 2. *Let X and Y be uniform spaces with the corresponding gages P and Q, respectively. Then each (P,Q)-isometry from X onto Y is uniformly continuous.*

Remark. Some authors call a map J isometry also if it maps X into Y. In this case our definition reduces on surjective isometry.

3. AN ESTIMATION FOR PSEUDOMETRIC CENTERS

Let M be a map from a set X into a set Y, endomed with the families of pseudometrics $\{d_i\}_{i\in I}$ and $\{D_j\}_{j\in J'}$, respectively. We introduce the following function defined for $t \geq 0$

(1) $$\phi_M^{i,j}(t) = \sup\{|D_j(M(x),M(y)) - d_i(x,y)| \;|$$
$$|d_i(x,y) \leq t \text{ or } D_j(M(x),M(y)) \leq t\}.$$

Definition 2. *The point* u_i *of the set X endowed with the family of pseudometrics* $\{d_i\}_{i\in I}$ *is called a* *__pseudometric center with respect to__* d_i *for* $x,y \in X$ *if there exists an isometry* $a_i : X \to X$ *such that* $a_i(x) = y$ *and for every* $v \in X$ *holds*

$$a_i(a_i(v)) = v, \quad d_i(v,a_i(v)) = 2d_i(v,u_i).$$

The point u of the set X is called a *__pseudometric center__* *for* $x,y \in X$ *if it is pseudometric center with respect to* d_i *for all* $i \in I$.

We have the following reformulation of the Proposition from [3].

Theorem 1. *Let* M *be a bijective map from* $(X,\{d_i\}_{i\in I})$ *onto* $(Y,\{D_j\}_{j\in J})$. *Let* ϕ *be any function such that* $\phi(t) \geq \phi_M^{i,j}(t)$ *for some* $i \in I$ *and some* $j \in J'$ *and* $\phi(2t) \leq 2\phi(t)$ *for every* t. *Let* $x,y \in X$ *and assume that this pair has a pseudometric center* u_i *with respect to* d_i *and that* $M(x)$, $M(y)$ *has a pseudometric center* v_j *with respect to* D_j. *If* n *be an integer such that* $2^n \geq (d_i(x,y)/(\phi(d_i(x,y)))$, *then*

$$D_j(v_j,M(u_i)) \leq 19\phi(d_i) + \phi(\tfrac{1}{2}d_i) + \phi(\tfrac{1}{4}d_i) + \dots + \phi(2^{-n}d_i), \tag{2}$$

where $d_i = d_i(x,y)$.

Theorem follows from the Proposition from [3] applying it on spaces (X,d_i) and (Y,D_j).

Definition 3. $(X,\{d_i\}_{i\in I})$ *has property* (P) *if for every* i, *every* $x,y \in X$ *and each* $r,t \geq 0$ *exists* $w_i \in X$ *such that* $d_i(x,w_i) \leq r$, *and* $d_i(y,w_i) \leq t$ *and* $d_i(x,y) = d_i(x,w_i) + d_i(y,w_i)$. *If the preceding is true for all* $i \in I$, *then we say the* X *has property* (P).

We remark, that from the preceding definition it follows: for each $x \in X$ and each $r \geq 0$ there exists $w \in X$ such that $d_i(x,w) \leq r$, i.e. each point x of X is an accumulation point.

Proposition 3. *Each locally convex space satisfies the condition* (P).

Proof. The element w from Definition 3. is of the form $sx + (1 - s)y$, $0 \leq s \leq 1$.

Proposition 4. *Let* M *be a map from a space* $(X,\{d_i\}_{i\in I})$ *onto space* $(Y,\{D_j\})_{j\in J}$ *both with property* (P). *Then for all* $i \in I$ *and all* $j \in J'$

$$\phi_M^{i,j}(r+t) \leq \phi_M^{i,j}(r) + \phi_M^{i,j}(t), \quad (r,t \geq 0).$$

Proof. We shall investigate separatelly two cases:
1) $d_i(x,y) \leq r+t$ and $D_j(M(x),M(y)) \geq d_i(x,y)$;

2) $D_j(M(x),M(y)) \le d_i(x,y)$ and $D_j(M(x),M(y)) \le r+t$.

In the first case, we take by (P) an element $w \in X$ such that $d_i(x,w) \le r$, $d_i(y,w) \le t$ and $d_i(x,y) = d_i(x,w) + d_i(y,w)$. Hence

$$|D_j(M(x),M(y)) - d_i(x,y)| = D_j(M(x),M(y)) - d_i(x,y)$$
$$\le D_j(M(x),M(w)) - d_i(x,w) - d_i(w,y) + D_j(M(w),M(y))$$
$$\le \phi_M^{i,j}(r) + \phi_M^{i,j}(t).$$

In the second case, there exists $w \in X$ such that $D_j(M(x),M(w)) \le r$, $D_j(M(w),M(y)) \le t$ and $D_j(M(x),M(y)) =$
$= D_j(M(x),M(y)) + D_j(M(w),M(y))$
(we have used that $M(X) = Y$ and the property (P) of Y). Hence

$$|D_j(M(x),M(y)) - d_i(x,y)| = d_i(x,y) - D_j(M(x),M(y))$$
$$\le d_i(x,w) - D_j(M(x),M(y)) + d_i(w,y) - D_j(M(w),M(y))$$
$$\le \phi_M^{i,j}(r) + \phi_M^{i,j}(t).$$

Theorem 2. *Let M be a bijective map from a locally convex space X, endowed with the family of seminorms $\{p_i\}_{i\in I}$, onto a sequentially complete Hausdorff space Y, endowed with the family of seminorms $\{P_j\}_{j\in J'}$. If for each $j \in J'$ there exists $i \in I$ such that*

(3) $$\int_1^\infty \frac{\phi_M^{i(j),j}(t)}{t^2} < \infty,$$

where

$$\phi_M^{i,j}(t) = \sup\{|P_j(M(x),M(y)) - p_i(x-y)| \mid p_i(x-y) \le t \text{ or } P_j(M(x) - M(y)) \le t\},$$

then there exists a linear (I,J')-isometry J from X onto Y such that

$$(4)\qquad P_j(M(x) - J(x)) \leq K\Big(p_{i(j)}(x) \int_{p_{i(j)}(x)}^{\infty} \frac{\phi^j(t)}{t^2}\, dt + \int_1^{p_{i(j)}(x)} \frac{\phi^j(t)}{t}\, dt\Big),$$

where K *is an universal constant and* $\phi^j(t) = \max\{1, \phi_M^{i(j),j}(t)\}$.

Proof. By Proposition 3. both X and Y have the property (P) and so by Proposition 4. it holds $\phi^{i,j}(2t) \leq 2\phi^{i,j}(t)$ where $\phi^{i,j}(t) = \max\{1, \phi_M^{i,j}(t)\}$. Now, we can apply Theorem 1. on the points $x = 0$ and $y = 2^n w$. The corresponding pseudometric centers are $u = (x+y)/2$ and $v = (M(x) + M(y))/2$ independent from i and j. So we have $u = 2^{n-1}w$ and $v = 2^{-1}M(2^n w)$. Let $M_n(w) := 2^{-n}M(2^n W) - 2^{-n+1}M(2^{n-1}w)$, $(n \in \mathbb{N})$.

If we suppose that $p_i(w) \leq 2^m$ for $m = 0,1,2,\ldots,$ then (2) implies

$$P_j(M_n(w)) \leq 2^{-n+1}(19\phi^j(2^{n+m}) + \phi^j(2^{n+m+1}) + \ldots$$
$$\ldots + \phi^j(1)).$$

Hence

$$(5)\qquad \sum_{n=1}^{\infty} P_j(M_n(w)) \leq 19 \sum_{n=1}^{\infty} 2^{-n+1}\phi^j(2^{n+m}) +$$
$$+ \sum_{j=m+1}^{\infty} \phi^j(2^j) \sum_{n=j-m}^{\infty} 2^{-n} + \sum_{j=0}^{m} \phi^j(2^j) \sum_{n=0}^{\infty} 2^{-n} <$$
$$< 20\cdot 2^m \sum_{n=m+1}^{\infty} 2^{-n}\phi^j(2^n) + 2\sum_{j=0}^{m}\phi^j(2^j).$$

Since (3) holds, we can use Lemma 2 from [3] and we obtain that

$$\sum_{n=m+1}^{\infty} 2^{-n}\phi^j(2^n)$$

converges. Then the preceding inequalities imply that $\sum_{n=1}^{\infty} P_j(M_n(w))$ converges for all $j \in J'$. Hence $\{\sum_{k=1}^{n} M_k(w)\}$ is a Cauchy sequence in X. Since Y is sequentialy complete, there exists $\sum_{n=1}^{\infty} M_n(w)$. Hence there exists

$$(6) \qquad \lim_{n\to\infty} 2^{-n}M(2^n w) = M(w) + \sum_{n=1}^{\infty} M_n(w),$$

which defines an operator $J : X \to Y$, i.e.

$$(7) \qquad J(w) = \lim_{n\to\infty} 2^{-n}M(2^n w).$$

(6) implies

$$P_j(J(x) - M(x)) \leq \sum_{n=1}^{\infty} P_j(M_n(w)).$$

By (5) and Lemma 2 from [3], taking m in the case $p_i(w) \geq 1$ such that $p_i(w) \geq 2^{m-1}$, we obtain (4).

We have to prove that the map J is a surjective linear (I,J')-isometry. We remark that condition (3) implies $\phi_M^{i(j),j}(t) = o(t)$ as $t \to \infty$.

Using this fact and Theorem 1. for $2^n x$, $2^n y$, $(x,y \in X;\ n \in \mathbb{N})$ we obtain for each $j \in J'$

$$P_j(2^{-n}M(2^n\cdot\tfrac{1}{2}(x+y)) - \tfrac{1}{2}(2^{-n}M(2^n x) + 2^{-n}M(2^n y))) \to 0$$

as $n \to \infty$.

Hence by (7)

$$(8) \qquad J(\tfrac{1}{2}(x+y)) = \tfrac{1}{2}(J(x) + J(y)),\ (x,y \in X).$$

Since

$$\phi_M^{i(j),j}(t) = \sup\{|P_j(M(x) - M(y)) - p_i(x-y)|\, p_i(x-y) \leq t$$

$$\text{or } P_j(M(x) - M(y)) \leq t\} = o(t) \text{ as } t \to \infty$$

and (7) hold, we obtain that for each $j \in J'$ there exists $i \in I$ such that

$$P_j(J(x) - J(y)) = p_i(x-y).$$

By (8) J is linear. We have to prove that $J(X) = Y$. Suppose that this is not true. Then there exists an element y from Y such that $P_j(y) = 1$ and $\mathcal{D}_j(y, J(X)) = \inf\{D_j(y,z); z\} \in J(x) \geq \frac{1}{2}$. Taking $x_n \in X$ such that $M(x_n) = ny$ we obtain $P_j(M(x_n) - J(x_n)) \geq \frac{n}{2}$. This is a contradiction with (4), since relation (4) implies $P_j(M(x_n) - J(x_n)) = o(n)$ as $n \to \infty$.

Combining the result of Gevirtz [1] and Theorem 2.1. from [5] we have:

Theorem 3. *Let* M *be a map from a real Banach space* X *onto a real Banach space* Y. *If for some* $\varepsilon > 0$ holds

$$| \|M(x) - M(y)\| - \|x-y\| | < \varepsilon, \quad (x,y \in X),$$

then M *is a* 15ε*-homomorphism, i.e.*

$$\|M(x+y) - M(x) - M(y)\| \leq 15\,\varepsilon.$$

REFERENCES

[1] Gevirtz, J.: *Stability of isometries on Banach spaces, Proc. Amer. Math. Soc.*,

[2] Hyres, D.M., Ulam, S.M.: *On approximate isometries, Bull. Amer. math. Soc. 51, (1945), 288-292.*

[3] Lindenstrauss, J., Szankowski, A.: *Non linear perturbations of isometries, Coll. Laurent Schwartz 1983, Vol. I, Asterisqe 131, (1985), 357-371.*

[4] Mazur, S., Ulam, S.: *Sur les transformations isométriques d'espaces vectoriels normes, Comptes Rendus Acad. Sci. Paris 194, (1932), 946-948.*

[5] Pap, E.: *Approximately and bounded homomorphisms on uniform commutative semigroups, Zbornik radova PMF u N. Sadu (to appear).*

[6] Rolewicz, S.: *Metric Linear Spaces*, *D. Reidel Publ. Comp.*, *1985*.

[7] Vogt, A.: *Maps which preserve equality of distance*, *Studia Math.* *45*, *(1973)*, *43-48*.

REZIME

BIJEKTIVNA PRESLIKAVANJA KOJA SU BLISKA PARCIJALNIM IZOMETRIJAMA

U radu je uveden pojam parcijalne izometrije sa skupa X na skup Y koji su snabdeveni familijama pseudometrike. Dokazano je da je za proizvoljno bijektivno preslikavanje M sa lokalno konveksnog vektorskog prostora na nizovno kompletan Hausdorffov lokalno konveksan vektorski prostor postoji parcijalna linearna izometrija koja je bliska preslikavanju M i data je procena za ovu bliskost.

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