Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 18,1,101-109(1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

NIKODYM TYPE THEOREM FOR METRIC VALUED x_0-EXHAUSTIVE SET FUNCTIONS

Endre Pap

University of Novi Sad, Faculty of Science,
Institute of Mathematics, Dr I. Djuričića 4,
21000 Novi Sad, Yugoslavia

ABSTRACT

In this paper the Nikodym theorem on uniform boundedness for x_0-exhaustive metric space valued set functions is proved.

The well-known Nikodym theorem on uniform boundedness of measures [8] has been generalized by many authors in different directions (see references). We can see all these aproachhes in the following simplified way. They consider a family F of set functions which are defined on a class of subsets of a given set (σ-ring, rings with some additional properties, etc.) with values in some topological algebraic structure (real numbers, Banach space, topological group, uniform semigroup, etc.) and this algebraic structure enables us to introduce some properties of set functions (additivity, subadditivity, k-triangularity, etc.) or combined with topology (countability, exhaustivity, etc.).

AMS Mathematical Subject Classification (1980): 28A33, 28B99.
Key words and phrases: x_0-exhaustive set function, d-bounded.

In this paper we shall consider set functions with values in a metric space M without supposing any algebraic structure on M. This implies that we have not any usual properties for set functions which would be implied with such an algebraic structure. We shall introduce the class of x_0-exhaustive set functions which are in connection with a fixed point x_0 from M. Using the main idea of I. Dobrakov [4], we have proved the uniform boundedness theorem for such a class of set functions

Let M be a metric space with the metric d.

We shall repeat the well-known

Definition 1. A subset A of a metric space (M,d) is d-bounded, or simply bounded, iff there exists $M' > 0$ such that

$$\sup_{x,y\in A} d(x,y) < M'.$$

We have the obvious

Proposition 1. The following statements are equivalent for a subset A of M:

a) There exists an element x_0 from M and a constant $M_{x_0} > 0$ such that

$$\sup_{y\in A} d(x_0,y) < M_{x_0};$$

b) For each element x from M there exists $M_x > 0$ such that holds

$$\sup_{y\in A} d(x,y) < M_x;$$

c) A is d-bounded.

Proof. a) $\Rightarrow$ b). Let for some $x_0 \in M$,

$$\sup_{y\in A} d(x_0,y) < M_{x_0}.$$

Then for an arbitrary fixed $x \in M$ it holds that

$d(x,y) \leq d(x,x_0) + d(x_0,y)$, $(y \in A)$. Hence,

$$\sup_{y\in A} d(x,y) < M'_{x_0},$$

where $M'_x = d(x,x_0) + M_{x_0}$. b) ⇒ c) and c) ⇒ a) are obvious.

Remark 1. In a topological vector space X a subset A is bounded (topological) if it is contained in all the sufficiently large multiples kV of any neigbourhood V of 0. If X is pseudometrizable, then a bounded set, in the previous sense, is also d-bounded. But the opposite is not true. For example, the whole vector space Z of all the sequences of complex numbers endowed with pointwise operations and metric

$$d(\{z_n\},\{w_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|z_n - w_n|}{1 + |z_n - w_n|}$$

is d-bounded, but not topologically bounded.
We shall need the following definition from [2].

Definition 2. A ring of sets Σ is called a quasi-σ-ring if any disjoint sequence in Σ possesses a subsequence which belongs to the family of disjoint sequences $\{A_n\}$ in Σ for which

$$\{ \bigcup_{n\in M} A_n \mid M \subset \mathbb{N}\} \subset \Sigma.$$

Let Σ be a quasi-σ-ring of sets and let M be a metric space.

Definition 3. A set function $\mu : \Sigma \to M$ is said to be x_0-exhaustive, for $x_0 \in M$, if

$$\lim_{n\to\infty} d(\mu(E_n),x_0) = 0$$

for each infinite sequence $\{E_n\}$ of pairwise disjoint sets from Σ.

Remark 2. If a set function $\mu : \Sigma \to M$ is x_0-exhaustive for $x_0 \in M$, then it cannot also be y_0-exhaustive for some

$y_0 \notin M$, $y_0 \neq x_0$.

Taking a special sequence of sets $\phi, \phi, \ldots$ in the preceding Definition 3, we obtain that for an x_0-exhaustive set function μ holds $\mu(\phi) = x_0$. This is the reason why in the case of the additive set functions with values in spaces with some algebraic structure, for example the topological group, we can consider only the 0-exhaustive set functions.

Proposition 2. Let $\mathcal{M}$ be a family of set functions $\mu : \Sigma \to M$. Then the set

$$\{\mu(E) \mid \mu \in \mathcal{M},\ E \in \Sigma\}$$

is d-bounded iff the following conditions hold:

(i) For each $m \in \mathbb{N}$ there exists $s(m) \in \mathbb{N}$ such that $d(\mu(A), \mu(B)) > s(m)$ implies
either $\sup\{d(\mu(C), x_0) \mid C \in \Sigma,\ C \subset A \setminus B\} > m$
or $\sup\{d(\mu(C), x_0) \mid C \in \Sigma,\ C \subset B \setminus A\} > m$,
$(\mu \in \mathcal{M};\ A, B \in \Sigma)$, and

(ii) the set $\{\mu(D_n) \mid \mu \in \mathcal{M},\ n \in \mathbb{N}\}$ is d-bounded for every sequence $\{D_n\}$ of pairwise disjoint sets from Σ.

Proof. The necessity of the conditions (i) and (ii) is obvious.

The conditions (i) and (ii) are sufficient. Namely, suppose that the set

$$\{\mu(E) \mid E \in \Sigma,\ \mu \in \mathcal{M}\}$$

is not d-bounded. Then for $x_0 \in M$ there exists a sequence $\{\mu_n\}$ from $\mathcal{M}$ and a sequence $\{E_n\}$ of sets from Σ such that

$$d(\mu_n(E_n), x_0) > n, \quad (n \in \mathbb{N}). \tag{1}$$

Condition (ii) implies $d(\mu_n(E_1), x_0) < M$ for some $M > 0$, so (1) implies

$$d(\mu_n(E_1), \mu_n(E_n)) \to \infty \quad \text{as } n \to \infty.$$

Hence, by (ii) either

a) $\sup\{d(\mu_n(C),x_0) \mid C \in \Sigma,\ C \subset E_n \setminus E\} \to \infty$

or

b) $\sup\{d(\mu_n(C),x_0) \mid C \in \Sigma,\ C \subset E_1 \setminus E_n\} \to \infty$.

Now, we shall choose a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ and a sequence of sets $\{D_n\}$ from Σ in the following way. Let $\mu_{n_1} = \mu_1$ and $D_1 = E_1$. In the case a) we choose n_2 such that $\mu_{n_2} = \mu_{i_2}$ where $i_2, i_3, \ldots$ is a sequence of natural numbers and $E_{i_2}, E_{i_3}, \ldots$ sets from $\{E_n\}$ and sets $E_j^1 \subset E_{i_j}$, $E_j^1 \in \Sigma$, $(j = 2,3,\ldots)$ such that

$$d(\mu_{i_j}(E_j^1 \setminus E_1),x_0) > j, \quad j = 2,3,\ldots .$$

We choose $D_2 = E_2^1 \setminus E_1$. Now, we repeat the procedure for sequences $\{\mu_{i_j}\}$ and $\{E_j^1 \setminus E_1\}_2^\infty$.

In the case b), we choose n_2 such that $\mu_{n_2} = \mu_{i_2}$ where we take that $i_2, i_3, \ldots$ is a sequence of natural numbers and $E_{i_2}, E_{i_3}, \ldots$ sets from $\{E_n\}$ and sets $E_j^1 \subset E_{i_j}$, $E_j^1 \in \Sigma$ $(j = 2,3,\ldots)$ such that

$$d(\mu_{i_j}(E_1 \setminus E_j^1),x_0) > j, \quad j = 2,3,\ldots .$$

Let $A_1 = E_1$ and $A_2 = E_1 \setminus E_2$.

Now, we shall repeat the procedure for the sequences $\mu_{i_2}, \mu_{i_3}, \ldots$ and $E_1 \setminus E_2^1,\ E_1 \setminus E_3^1, \ldots$.

Repeating these procedures we either construct a sequence $\{\mu_{n_k}\}$ of set functions from $\mathcal{M}$ and a sequence $\{D_k\}$ of pairwise disjoint sets from Σ such that

$$d(\mu_{n_k}(D_k),x_0) > k,\ (k \in \mathbb{N}),\ \text{(contradiction with (ii)!)},$$

or we construct a sequence $\{\mu_{n_k}\}$ of set functions from $\mathcal{M}$ and a non-increasing sequence $\{A_k\}$ of sets from Σ such that

$$d(\mu_{n_k}(A_k),x_0) > k, \quad (k \in \mathbb{N}).$$

Then condition (i) implies the existence of subsequences $\{\mu_{n_{k_i}}\}$ and $\{A_{k_i}\}$ such that

$$d(\mu_{n_{k_i}}(A_{k_{i-1}} \setminus A_{k_i}), x_0) > i, \quad (i \in \mathbb{N})$$

for $k_0 = 1$, which is in contradiction with (ii).

We have the obvious

Proposition 3. Let $\mu : \Sigma \to M$ be an x_0-exhaustive set function, and let $\{A_n\}$ be a sequence of pairwise disjoint elements from Σ. Then for each $\varepsilon > 0$, there exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that

$$\sup\{d(\mu(C), x_0) \mid C \in \Sigma, \; C \subset \bigcup_{i \in I} A_{n_i}\} < \varepsilon,$$

for any $I \subset \mathbb{N}$.

Now, we have the main result

Theorem. Let $\mathcal{M}$ be a family of x_0-exhaustive set functions $\mu : \Sigma \to M$, where M is a metric space.

Then the set $\{\mu(E);\ \mu \in \mathcal{M},\ E \in \Sigma\}$ is d-bounded iff the following conditions hold

(i) for each $m \in \mathbb{N}$ there exists $s(m) \in \mathbb{N}$ such that $d(\mu(A), \mu(B)) > s(m)$, implies
either $d(\mu(A \setminus B), x_0) > m$,
or $d(\mu(B \setminus A), x_0) > m$

and

(ii) the set

$$\left\{ \lim_{n\to\infty} \sup\{d(\mu(A \cup B), x_0) \mid B \in \Sigma, \; \sup\{d(\mu(C), x_0) \mid C \in \Sigma, \; C \subset B\} < \frac{1}{n}\} \mid \mu \in \mathcal{M} \right\}$$

is bounded for each $A \in \Sigma$.

Proof. Conditions (i) and (ii) are obvious and necessary. Suppose now that conditions (i) and (ii) hold but the set

$$\{\mu(E);\ \mu \in \mathcal{M},\ E \in \Sigma\}$$

is not d-bounded. Then by Proposition 1 and Proposition 2 there exist a sequence $\{\mu_n\}$ from $\mathbf{M}$ and a sequence $\{D_n\}$ of pairwise disjoint sets from Σ such that

$$d(\mu_n(D_n),x_0) > n, \quad (n \in \mathbb{N}).$$

We take that

$$(2) \qquad m_1 = [\sup_n\{\lim_{s\to\infty} \sup\{d(\mu_n(D_1 \cup B),x_0) \mid B \in \Sigma, \sup_{\substack{C\subset B\\ C\in\Sigma}} d(\mu_n(C),x_0)<\tfrac{1}{s}\}\}] + 1 \in \mathbb{N}, \text{ by (ii)}.$$

Then, there exists $k(m_1) \in \mathbb{N}$, by (i). We take that $n_1 > s(m_1) + 1$. By Proposition 3 and (2) there exists a subsequence $\{D_n^1\}$ of the sequence $D_{n_1+1}, D_{n_2+2}, \ldots$ such that

$$d(\mu_{n_1}(D_1 \cup \bigcup_{i\in I} D_i^1) < m_1$$

for arbitrary $I \subset \mathbb{N}$. Now, we shall take that

$$m_2 = [\sup_n\{\lim_{s\to\infty} \sup\{d(\mu_n(D_1 \cup D_{n_1} \cup B)\}x_0) \mid B \in \Sigma, \sup_{\substack{C\subset B\\ C\in\Sigma}} d(\mu_n(C),x_0)<\tfrac{1}{s}\}\}] + 1 \in \mathbb{N},$$

by (ii) and repeating the preceding procedure. Continuing this procedure we can construct two sequences of natural numbers $\{m_k\}$ and $\{n_k\}_0^\infty$, $n_0 = 1$, such that

$$m_k = [\sup_n\{\lim_{s\to\infty} \sup\{d(\mu_n(\bigcup_{i=0}^{k-1} D_{n_i} \cup B),x_0) \mid B \in \Sigma, \sup_{\substack{C\subset B\\ C\in\Sigma}} d(\mu_n(C,x_0) < \tfrac{1}{s}\}\}] + 1 \in \mathbb{N},$$

$$(3) \qquad d(\mu_{n_k}(D_{n_k}),x_0) > n_k > s(m_k) + k$$

and

$$(4) \qquad d(\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i} \setminus D_{n_k}),x_0) < m_k, \quad (k \in \mathbb{N}).$$

If we take that $k > \sup_j d(\mu_{n_j}(\bigcup_{i=0}^{\infty} D_{n_i}), x_0)$, then we obtain by (3) that

$$d(\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i}), \mu_{n_k}(D_{n_k})) \geq d(\mu_{n_k}(D_{n_k}), x_0) - d(\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i}), x_0) >$$

$$> s(m_k).$$

By (i), this implies

$$d(\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i} \setminus D_{n_k}), x_0) > m_k, \quad (k \in \mathbb{N}),$$

a contradiction with (4).

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REZIME

NIKODYMOVA TEOREMA ZA x_0-EKSHAUSTIVNE SKUPOVNE FUNKCIJE SA VREDNOSTIMA U METRIČKOM PROSTORU

Za kvazi-σ-prsten Σ i metrički prostor M u radu se uvodi pojam x_0-ekshaustivne skupovne funkcije $\mu : \Sigma \to M$, za $x_0 \in M$, - Definicija 3. Za ovakve skupovne funkcije se dokazuje teorema tipa Nikodyma o uniformnoj ograničenosti. Na taj način se karakterišu, u smislu ograničenosti, i familije skupovnih funkcija sa vrednostima u metričkom prostoru M, bez pretpostavke o postojanju algebarske operacije u M, a time i uobičajenih osobina skupovnih funkcija.

Received by the editors December 2, 1987.