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A UNIFORMLY CONVERGENT SPLINE DIFFERENCE SCHEME FOR A SELF-ADJOINT SINGULAR PERTURBATION PROBLEM

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ABSTRACT

For the problem: $-\varepsilon y''+q(x)y = f(x)$, $0<x<1$, $y(0) = a_0$, $y(1) = a_1$, the exponentially fitted spline difference scheme is derived. This scheme has the second order of uniform accuracy under some conditions on the functions $q(x)$ and $f(x)$.

1. INTRODUCTION

We shall consider the problem

$$Ly = -\varepsilon y'' + q(x)y = f(x), \qquad 0<x<1,$$

(1) $$y(0) = a_0, \qquad y(1) = a_1,$$

$$q(x) \geq q > 0, \qquad 0 < \varepsilon \leq 1.$$

The cubic spline difference scheme for problem (1) is derived in [3], [4]. In [3] we required that the spline

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$v(x)\in C^2[0, 1]$ satisfies the differential equation

$$-\sigma(x)v''(x) + q(x)v(x) = f(x),$$

at the points x_i, $x_i = x_{i-1} + h$, $i = 1(1)n + 1$, $x_o = 0$, $x_{n+1} = 1$, where $\sigma(x)$ is an exponential fitting factor. In this way the difference scheme uniformly stable in ε has obtained. That scheme has the form

(2) $$R_h v_i = Q_h f_i,\ i = 1(1)n, \quad v_i = v(x_i), \quad f_i = f(x_i),$$

$$R_h v_i = -r^- v_{i-1} + r^c v_i - r^+ v_{i+1},$$

$$Q_h f_i = q^- f_{i-1} + q^c f_i + q^+ f_{i+1}, \quad \text{where}$$

$$r^- = (1 - \frac{h^2 q_{i-1}}{6\ \sigma_{i-1}})\frac{1}{h}, \quad r^c = 2(1 + \frac{h^2 q_i}{3\ \sigma_i}) \cdot \frac{1}{h},$$

$$r^+ = (1 - \frac{h^2 q_{i+1}}{6\ \sigma_{i+1}}) \cdot \frac{1}{h}, \quad q^- = \frac{h}{6\ \sigma_{i-1}}, \quad q^c = \frac{2h}{3\ \sigma_i}$$

$$q^+ = \frac{h}{6\ \sigma_{i+1}}, \quad v_o = a_o, \quad v_{n+1} = a_1, \quad \sigma_i = \sigma(x_i).$$

In this paper we shall determine σ_i in such way that scheme (2) becomes a uniformly convergent one in ε.

2. DETERMINATION OF THE FITTING FACTOR

In order to get a suitable fitting factor σ_i we shall use the following lemma.

Lemma 1, [1]. *Let* $y(x)\in C^4[0, 1]$. *Let* $q'(0) = q'(1) = 0$. *Then the solution of problem (1) has the form*

(3) $$y(x) = u_o(x) + w_o(x) + g(x), \text{ where}$$

$$u_o(x) = p_o \exp(-x\sqrt{q(0)/\varepsilon}),$$

$w_o(x) = p_1\exp(-(1-x)\sqrt{q(1)/\varepsilon})$, p_o *and* p_1 *are bounded functions of* ε *independent of* x *and,*

(4) $$|g^{(i)}(x)| \le M(1 + \varepsilon^{1+\frac{1}{2}i}),\ i=0(1)4,$$

M *is a constant independent of* ε.

If we suppose that $q(x) \equiv q$ = const. and require that $R_h u_o(x_i) = 0$, we obtain that

$$\sigma = \frac{h^2 q}{6}\left(1 + \frac{3}{2\,\mathrm{sh}^2\rho}\right), \qquad \rho = \frac{h}{2}\sqrt{q/\varepsilon}.$$

For the same σ we have $R_h w_o(x_i) = 0$. Now, we shall define σ_i for the case $q(x) \neq$ const., as

(5) $$\sigma_i = \frac{h^2 q_i}{6}\left(1 + \frac{3}{2\,\mathrm{sh}^2\rho_i}\right), \qquad \rho_i = \frac{h}{2}\sqrt{q_i/\varepsilon}.$$

Throughout the paper M denotes different constants independent of ε and h. From (5) we see that $0 \le \sigma_i \le Mh^2$, for $\varepsilon \le h^2$. Because of this, we have

$$|\sigma_i - \varepsilon| \le Mh^2, \text{ for } \varepsilon \le h^2.$$

That the same holds if $h^2 \le \varepsilon$, we can see from the fact

$$|\rho_i^2 \mathrm{sh}^{-2}\rho_i - 1| \le M\rho_i^2.$$

Thus,

(6) $$|\sigma_i - \varepsilon| \le Mh^2.$$

3. THE TRUNCATION ERROR

The truncation error for the arbitrary smooth function $r(x)$ is defined as

$$\tau_i(r) = R_h r_i - Q_h(Lr(x))_i .$$

Because of (3), we have

(7) $$\tau_j(y) = \tau_j(u_o) + \tau_j(w_o) + \tau_j(g) = R_h y_j - Q_h(Ly)_j =$$

$$= R_h y_j - R_h v_j = R_h z_j, \qquad j = 1(1)n,$$

$$z_j = z(x_j), \quad z(x) = y(x) - v(x).$$

By the procedure which is applied in [2] or [3], we have

(8) $$\tau_i(y) = R_h z_i,$$

$$\tau_i(y) = \frac{1}{h}(\varphi_{2,i+1}(y) - \varphi_{2,i}(y)) + \varphi_{1,i}(y),$$

$$\varphi_{1,i}(y) = \psi_i^{(1)} - h\frac{\psi_i^{(2)}}{2} + \frac{h}{2}\left(\frac{\eta_{i-1}}{\sigma_{i-1}} + \frac{\eta_i}{\sigma_i}\right),$$

$$\eta_i = y_i''(\sigma_i - \varepsilon),$$

$$\psi_i^{(k)} = \frac{h^{4-k}}{(4-k)!}\, y^{IV}(\xi_i), \quad x_{i-1} \le \xi_i \le x_i, \quad (k=0,1,2)$$

$$\varphi_{2,i}(y) = \psi_i^{(0)} + h^2\left(\frac{\eta_{i-1}}{3\,\sigma_{i-1}} + \frac{\eta_i}{6\,\sigma_i} - \frac{\psi_i^{(2)}}{6}\right).$$

According to Lemma 1 and (7), we shall estimate separately the truncation error for $u_o(x)$, $w_o(x)$ and $g(x)$. Starting with $u_o(x)$, we consider first the case $h^2 \le \varepsilon$.

Let $\bar{\tau}_i(u_o)$ be the truncation error when $q(x)=q=\text{const}$. Then, $R_h u_o(x_i)=0$, $Lu_o = 0$ and $\bar{\tau}_i(u_o) = 0$. Thus, from (8) we have

$$(9)\qquad \tau_i(u_o) = \tau_i(u_o)-\bar{\tau}_i(u_o) = \frac{1}{h}(\varphi_{2,i+1}(u_o)-\bar{\varphi}_{2,i+1}(u_o) - $$
$$- \varphi_{2,i}(u_o)+\bar{\varphi}_{2,i}(u_o))+\varphi_{1,i}(u_o)-\bar{\varphi}_{1,i}(u_o),$$

where $\bar{\varphi}_{2,i}$ and $\bar{\varphi}_{1,i}$ are functions $\varphi_{2,i}$ and $\varphi_{1,i}$ in the case $q(x) = q(0) = q_o$

$$\varphi_{1,i}-\bar{\varphi}_{1,i} = \frac{\varepsilon h}{2}\, u''_o(x_{i-1})(\sigma_o^{-1}-\sigma_{i-1}^{-1}) +$$
$$+ \frac{\varepsilon h}{2}\, u_o(x_i)(\sigma_o^{-1}-\sigma_i^{-1}).$$

Since, $\sigma_i = \varepsilon + O(h^2)$, $i=0(1)n$, and $|q_i-q_o| \le M\, x_i^2$ after some Taylor developments, we have $|\sigma_o^{-1} - \sigma_i^{-1}| \le M\, h^2\varepsilon^{-2}x_i^2$, and

$$|\varphi_{1,i}-\bar{\varphi}_{1,i}|\le M(\frac{x_{i-1}^2h^3}{\varepsilon^2}\exp(-x_{i-1}\sqrt{q_o/\varepsilon}) +$$
$$+ \frac{x_i^2h^3}{\varepsilon^2}\exp(-x_i\sqrt{q_o/\varepsilon}))\le Mh^3\varepsilon^{-1}\exp(-x_i\delta\sqrt{q_o/\varepsilon}),$$

where δ is a constant independent of ε and h. In the same way, we have that $|\varphi_{2,i}-\bar{\varphi}_{2,i}| \le M\, h^4\varepsilon^{-1}\exp(-x_i\delta\sqrt{q_o/\varepsilon})$, and from (9) we have

$$(10)\qquad |\tau_i(u_o)| \le M\, h^3\varepsilon^{-1}\exp(-x_i\delta\sqrt{q(0)/\varepsilon}).$$

Analogously, we have

$$(11)\qquad |\tau_i(w_o)| \le M\, h^3\varepsilon^{-1}\exp(-(1-x_i)\delta\sqrt{q(1)/\varepsilon}).$$

Further,

(12) $|\tau_i(g)| \le M\, h^3\varepsilon^{-1}$, because of

$$|\eta_i| = g''_i(\sigma_i-\varepsilon)| \le M\, h^2,\ |\psi_i^{(k)}| \le M\, \varepsilon^{-1}h^{4-k}.$$

Finally, from (10), (11), (12) and (7), we have

(13) $\quad |\tau_i(y)| \le M\, h^3\varepsilon^{-1} \quad$ when $\quad h^2 \le \varepsilon.$

Let $\varepsilon \le h^2$. Then $\sigma_i = 0(h^2)$ and

$$|\sigma_o^{-1}-\sigma_i^{-1}| \le M|q_i-q_o+\frac{3}{2}(q_i-q_o)\omega_i+\frac{3}{2}\,q_o(\omega_i-\omega_o)|,$$

where $\omega_i = sh^2\rho_i$.

$$|\sigma_o^{-1}-\sigma_i^{-1}| \le M\, h^{-2}(x_i^2+x_i^2 h\,\varepsilon^{-\frac{1}{2}}\, sh^{-3}(\xi)ch(\xi)),$$

ξ is a point between ρ_o and ρ_i. Since $x_{i-1} = 0$ for i=1, from (8) we have

$$|\varphi_{1,i}(u_o)-\bar{\varphi}_{1,i}(u_o)| \le Mh\ \exp(-x_{i-1}\delta\sqrt{q_o/\varepsilon}),\ i=1(1)n,$$

and

(14) $\quad |\tau_i(u_o)| \le Mh\ \exp(-x_{i-1}\delta\sqrt{q(0)/\varepsilon})$

Similarly,

(15) $\quad |\tau_i(w_o)| \le Mh\ \exp(-(1-x_{i+1})\delta\sqrt{q(1)/\varepsilon})$

$$\tau_i(g) = T_o g_i+T_1 g_i' - \frac{1}{2}(r^-h^2g''(b_1)+r^+h^2g''(b_2)) -$$

$$- \frac{1}{2}(q^-q_{i-1}h^2g''(b_1)+q^+q_{i+1}h^2g''(b_2))-\varepsilon(q^-g_{i-1}'' +$$

$$+ q^+g_{i+1}''+q^cg_i''),\quad x_{i-1}\le b_1\le x_i\le b_2\le x_{i+1}.$$

Since, $|g''(x)|\le M$ and $|r^-|\le Mh^{-1}$, $|r^+|\le Mh^{-1}$, $|q^-|\le Mh^{-1}$, $|q^+|\le Mh^{-1}$, $T_o=T_1=0$, we have $|\tau_i(g)|\le Mh$, and finally ((15), (14)),

(16) $\quad |\tau_i(y)| \le Mh \quad$ when $\quad \varepsilon \le h^2.$

Proof of the uniform convergence. Denote by A the matrix of system (2). Then, from (8), we obtain

(17) $$|y_i - v_i| \le ||A^{-1}|| \max_i |\tau_i(y)|.$$

Since $||A^{-1}|| \le \max_i |-r^- + r^c - r^+|^{-1} \le M \max_i \sigma_i h^{-1} \le$

$$\le \begin{cases} M\varepsilon h^{-1}, & h^2 \le \varepsilon, \\ Mh, & \varepsilon \le h^2, \end{cases}$$

from (16, (13) and (17), we have

$$|y_i - v_i| \le Mh^2.$$

Thus, the following theorem holds.

Theorem 1. *Assume* $q(x) \ge q > 0$, $q'(0) = q'(1) = 0$ *and* $f(x)$, $q(x) \in C^2[0, 1]$ *in* (1). *Let* v_i, $i = 0(1)n + 1$, *be the approximation to the solution* $y(x)$ *of* (1) *obtained using* (2). *Then, there is a constant* M *independent of* ε *and* h, *such that*

$$|y(x_i) - v_i| \le Mh^2, \quad i = 0(1)n + 1.$$

4. A NUMERICAL EXPERIMENT

Our test for the order of uniform convergence, notation and example are taken from [1].

$$-\varepsilon y'' + (1+x^2)^2\, y = 4(3x^2-3x+1)(1+x)^2$$

$$y(0) = -1, \; y(1) = 0$$

ε \ k	0	1	2	3	4	$\bar{p}_{\varepsilon}$
1/2	2.00	2.00	2.00	2.00	2.00	2.00
1/4	2.00	2.00	2.00	2.00	2.00	2.00
1/8	2.03	2.00	2.00	2.00	2.00	2.00
1/16	2.00	2.02	2.00	2.00	2.00	2.00
1/32	2.08	2.03	2.00	2.00	2.00	2.02
1/64	1.86	2.06	2.01	2.00	2.00	2.01
1/128	1.24	2.08	1.99	2.01	2.00	1.86
1/256	0.85	1.69	2.06	2.02	2.00	1.72
1/512	1.36	0.97	2.08	1.99	2.01	1.68

The computed order of uniform convergence is 1.68 and the clasical one is 2.00, $q'(0) \neq q'(1)$.

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REZIME

UNIFORMNO KONVERGENTNA SPLAJN DIFERENCNA ŠEMA ZA SAMO-ADJUNGOVANI SINGULARNO PERTURBACIONI PROBLEM

Za problem: $-\varepsilon y''+q(x)\,y=f(x)$, $0<x<1$, $y(0)=a_0$, $y(1)=a_1$, izvedena je eksponencijalno "fitovana" splajn diferencna šema. Šema ima drugi red uniformne tačnosti pod odredjenim uslovima na funkcije $q(x)$, $f(x)$.

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