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ON COVARIANTLY-PROJECTIVE TRANSFORMATIONS OF SELF-RECURRENT SW-0_n

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ABSTRACT

The subject of the paper is a self-recurrent SW-0_n and its covariantly geodesic lines; first, conditions are found for such a SW-0_n to be π-projective to its own adjoint Riemannian space; then, conditions are found for two self-recurrent SW-0_ns to be covariantly projective, that is, to have covariantly geodesic lines in common; third, it is proved that two different SW-0_ns over the same Riemannian space cannot be π-projective.

1. INTRODUCTION

Let us consider an n-dimensional space V_n^Γ with an object of a linear connection $\{\Gamma^i_{jk}\}$, provided by a tensor field (π_{ij}) of type (0,2), which is regular.

By the notation of K. Radiszewski, a vector field w is π-geodesic if

$$(0.1)\qquad \nabla_k(\pi_{ia}w^a)w^k = \lambda\pi_{ia}w^a$$

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where ∇_k denotes the covariant differentiation regarding the connection Γ.

A π-geodesic line is an integral curve of a π-geodesic vector field.

We shall use the fact, that π-geodesics in V_n^Γ are geodesics in the usual sense in the space V_n^G, where G is given by

$$(0.2)\qquad G_{jk}^i = (\nabla_j \pi_{pk})\bar{\pi}^{pi} + \Gamma_{jk}^i$$

$\bar{\pi}^{pi}$ denotes the inverse of the tensor (π_{ij}), which is regular.

The connections Γ and $\bar{\Gamma}$ have their π-geodesics in common if and only if the connections G and $\bar{G}$ have their geodesics in common, that is, if

$$(0.3)\qquad \bar{\Gamma}_{jk}^i = \Gamma_{jk}^i - \pi_{jk}\psi_s\bar{\pi}^{si} - \psi_j\delta_k^i$$

where ψ_j is an arbitrary vector field. Then we say that the spaces V_n^Γ and $V_n^{\bar{\Gamma}}$ are π-projective.

Let us presume one more condition: for the tensor field (π_{ij}) to be symmetric. If we want both Γ and G to be symmetric, we have to involve the next condition

$$(0.4)\qquad \nabla_k\pi_{ij} = \nabla_j\pi_{ik}.$$

Now, let us consider a Riemannian space M_n^g, with metric tensor g and a regular linear isomorphism of M_n^g into itself, namely (P_j^i).

We define the regular general covariant differentiation for a tensor field T_{jk}^i

$$(0.5)\qquad T_{jk,\ell}^i = P_a^i T_{bc|\ell}^a P_j^b P_k^c,$$

where $T^{a}_{bc|\ell}$ is the basic covariant derivative. It can be obtained with two affine connections, $'\Gamma$ and $''\Gamma$. $'\Gamma$ works exceptionally on contravariant indices and $''\Gamma$ works on covariant ones. The additional condition is that the basic covariant derivative of the tensor Q (the inverse of the tensor P) vanishes. M^{g}_{n} with the tensor P and regular general covariant differentiation is a space of a regular general connection.

A space of a regular general connection is a SW-O_n if it is a metric space and if the following conditions hold:

a) $g_{ij,k} = \gamma_k m_{ij}$ (γ_k is a vector field and m_{ij} is a symmetric tensor field),

b) the connection $'\Gamma$ is symmetric,

c) $P_{ij} = g_{ia}P^{a}_{j}$ is symmetric.

For further properties, the reader can see [1], [3], [4].

The underlying Riemannian geometry is called the adjoint Reimannian space of SW-O_n. Its covariant differentiation (Levi-Civita covariant differentiation) is denoted by ∇ (usually $\overset{0}{\nabla}$ in previous papers).

We are especially interested in such classes of SW-O_ns, where γ_k from a) is equal to zero. We denote $'\Gamma$ and $''\Gamma$ in such a case by $'\overset{m}{\Gamma}$ and $''\overset{m}{\Gamma}$. One could prove that $''\overset{m}{\Gamma}$ is in fact a metric affine connection, but $'\overset{m}{\Gamma}$ is not.

In two previous papers ([4], [5]), we considered such a special tensor P, that

$$(0.6)\qquad \nabla_k P^{i}_{j} = \alpha_k \delta^{i}_{j}$$

SW-O_n, defined by such a tensor P and γ_k from a) equal to zero, we named it a self-recurrent SW-O_n. Its connection $''\overset{m}{\Gamma}$ is a semi--symmetric metric connection :

$$(0.7)\qquad {}^{m}\Gamma^{i}_{jk} = \{{}^{\,i}_{jk}\} + \tilde{a}_j\delta^i_k - g_{jk}\tilde{a}^i$$

where $\tilde{a}_j$ means the image of α_j, by the isomorphism Q.

1. THE POSSIBILITY OF π-PROJECTIVITY OF THE ADJOINT RIEMANNIAN SPACE TO ITS SELF-RECURRENT SW-O_n

Suppose we have found the geodesics for the parallel displacement defined by the connection (0.7), in the ordinary sense. We shall call them the covariantly autoparallel lines of SW-O_n. Now, we shall find the condition for such lines to be π-geodesics in the adjoint Riemannian space; (π_{ij}) is any regular tensor field of type (0.2) on M^g_n; suppose, also, that it is symmetric.

From (0.2), we get

$$(1.1)\qquad (\nabla_j\pi_{ps})\overset{-1}{\pi}{}^{ip} = \tilde{a}_j\delta^i_s - g_{sj}\tilde{a}^i$$

or

$$(1.2)\qquad \nabla_j\pi_{ks} = \tilde{a}_j\pi_{ks} - g_{sj}\pi_{ki}\tilde{a}^i \;.$$

Since we have no more presumptions about the tensor (π_{ij}) but its regularity and symmetry, its Levi-Civita covariant derivative also has to be symmetric:

$$(1.3)\qquad \nabla_j\pi_{ks} - \nabla_j\pi_{sk} = 0$$

and, after calculation

$$\tilde{a}^i(g_{kj}\pi_{si} - g_{sj}\pi_{ki}) = 0 .$$

Raising the index j, we get

$$(1.4)\qquad \tilde{a}^i(\delta^j_k\pi_{si} - \delta^j_s\pi_{ki}) = 0$$

and, contracting the indices j and k

(1.5) $\tilde{a}^i \pi_{is} = 0$

which means that the image of the vector $\tilde{a}^i$ by the transformation (π_{ij}) is zero. Now, we have two possibilities.

(1) If (π_{ij}) is regular, its kernel is zero-vector, so $\tilde{a}^i$ is zero. Since (Q_{ij}) is also regular, its kernel is also zero. Under such a condition, (0.7) is just a Levi-Civita connection and SW-O_n is trivial.

(2) If $(\tilde{a}^i)$ is not a zero-vector, it is an element of the kernel of (π_{ij}) and, consequently, (π_{ij}) is not an isomorphism, i.e. it is not regular.

(1) and (2) give us the next result:

Lemma 1. *If covariantly autoparallel lines of self-recurrent* SW-O_n *should be* π*-geodesics of its adjoint Riemannian space, then the tensor field* (π_{ij}) *of type (0.2) should not be symmetric.*

Now, we have to check if the tensor field (π_{ij}) could be skew-symmetric.

Then, if it is, its covariant derivative is also skew-symmetric in indices k and s, namely

(1.6) $\nabla_j \pi_{ks} + \nabla_j \pi_{sk} = 0$

and, after calculation

$$(g_{sj}\pi_{ki} + g_{kj}\pi_{si})\tilde{a}^i = 0 .$$

After raising the index j, we get

$$(\delta_s^j \pi_{ki} + \delta_k^j \pi_{si}) \tilde{a}^i = 0 .$$

Contracting the indices j and k, we can get

(1.7) $$\pi_{si}\tilde{a}^i = 0$$

and, then, we can state

Lemma 2. *If covariantly autoparallel lines of self-recurrent SW-O_n should be π-geodesics of the adjoint Riemannian space, then the tensor field (π_{ij}) of type (0.2) should not be skew-symmetric.*

Now, we know that, if covarianlty autoparallel lines of self-recurrent SW-O_n should be π-geodesics of its adjoint Riemannian space, then the tensor field (π_{ij}) must be neither symmetric nor skew-symmetric. Then the difference $\pi_{ks} - \pi_{sk}$ is equal neither to $2\pi_{ks}$ nor to zero. We shall denote this difference by T_{ks}. It is a regular skew-symmetric tensor field of type (0.2). Moreover, since T_{ks} is skew-symmetric, then

(1.8) $$\nabla_j T_{ks} + \nabla_j T_{sk} = 0$$

and

$$\nabla_{(j}T_{ks)} = 0$$

which means that (T_{ks}) is a Killing tensor field. Then we have

Lemma 3. *If covariantly autoparallel lines of self-recurrent SW-O_n should be π-geodesics of its adjoint Riemannian space, then the adjoint Riemannian space has to admit at least one Killing tensor field of order two, which is not equal to zero.*

For T_{ks}, we have

$$\nabla_j T_{ks} = \tilde{\alpha}_j T_{ks} - g_{sj}\pi_{ki}\tilde{\alpha}^i + g_{kj}\pi_{si}\tilde{\alpha}^i$$

and, as it is a Killing tensor field

$$g^{aj}\nabla_j T_{ka} = 0$$

which gives us

$$T_{ka}\tilde{\alpha}^a - n\pi_{ki}\tilde{\alpha}^i + \pi_{ki}\tilde{\alpha}^i = 0$$

i.e.

$$(1.9)\qquad T_{ka}\tilde{\alpha}^a = (n-1)\pi_{ki}\tilde{\alpha}^i$$

and $n \neq 2$.

On the other hand, $\nabla_{[j}T_{ks]} = \nabla_j T_{ks}$, by the reason of T_{ks} being a Killing tensor field. That means

$$\tilde{\alpha}_j T_{ks} - g_{sj}\pi_{ki}\tilde{\alpha}^i + g_{kj}\pi_{si}\tilde{\alpha}^i = \frac{1}{3}(\tilde{\alpha}_j T_{ks} + \tilde{\alpha}_k T_{sj} + \tilde{\alpha}_s T_{jk}).$$

Transvecting by $g^{j\ell}$ and contracting the indices ℓ and s, we get again (1.9). But, (1.9) means

$$\pi_{ka}\tilde{\alpha}^a - \tilde{\alpha}^a\pi_{ak} = (n-1)\pi_{ki}\tilde{\alpha}^i$$

and, consequently

$$(1.10)\qquad (2-n)\pi_{ka}\tilde{\alpha}^a = \tilde{\alpha}^a\pi_{ak}\,.$$

In this way, we got a characterization of non-symmetry of the tensor field (π_{ij}).

We have

Theorem 1. *If the covariantly autoparallel lines of self-recurrent* SW-O_n *are* π*-geodesics of its adjoint Riemannian space, then the tensor field* (π_{ij}) *is a non-symmetric tensor* *field and*

$$(2-n)\pi_{ka}\tilde{\alpha}^a = \tilde{\alpha}^a\pi_{ak}$$

for the vector field $\tilde{\alpha}^i$ *characterizing the self-recurrent* SW-O_n.

If the adjoint Riemannian space is of constant negative curvature, then there is no Killing tensor of order p (p=1, 2,...,n-1) ([6]) and covariantly autoparallel lines cannot be π-geodesics in the adjoint Riemannian space in any way.

Also, in the conformally flat Riemannian space with a negative definite Ricci curvature, there does not exist any Killing tensor different from zero. So, if the adjoint Riemannian space is conformally flat and has a negative definite Ricci curvature, there could not be any π-projectivity between it and its self-recurrent SW-O_n.

2. THE POSSIBILITY OF COVARIANT PROJECTIVITY OF TWO DIFFERENT SELF-RECURRENT SW-O_ns

Let us suppose we are given two different regular tensor fields of type (0.2), namely (A_{ij}) and (B_{ij}), both recurrent to Kronecker's delta, in M_n^g:

(2.1) $$\nabla_k A_j^i = \alpha_k \delta_j^i \qquad (\nabla_k A_{ij} = \alpha_k g_{ij})$$

(2.2) $$\nabla_k B_j^i = \beta_k \delta_j^i \qquad (\nabla_k B_{ij} = \beta_k g_{ij}) .$$

Each tensor field (A_{ij}), (B_{ij}) defines uniquely a self-recurrent SW-O_n, both of them having M_n^g as a common adjoint Riemmannian space. Let us denote by $(\tilde{A}_{ij})$ and $(\tilde{B}_{ij})$ the inverses of (A_{ij}) and (B_{ij}) respectively. Following the way of notation from the previous paragraphs, we shall denote

(2.3) $$\tilde{\alpha}_k = \alpha_a \tilde{A}_k^a$$

$$(2.4) \qquad \tilde{\beta}_k = \beta_a \tilde{B}^a_k$$

by the same overbar ~, although these are two different isomorphisms. By (0.7), we have

$$(2.5) \qquad {}^{m}_{1}\Gamma^{i}_{jk} = \left\{ {i \atop jk} \right\} + \tilde{\alpha}_j \delta^i_k - \tilde{\alpha}^i g_{jk}$$

$$(2.6) \qquad {}^{m}_{2}\Gamma^{i}_{jk} = \left\{ {i \atop jk} \right\} + \tilde{\beta}_j \delta^i_k - \tilde{\beta}^i g_{jk}$$

We have defined covariantly autoparallel lines of a self-recurrent SW-O_n as geodesics, in the ordinary sense, of the connection (0.7) ((2.5), (2.6)). Now, we want to find the conditions for two different self-recurrent SW-O_ns over the same Riemannian space to have their covariantly autoparallel lines in common.

It is well-known that the change of the connection which does not change the system of geodesics can be expressed locally in this way:

$$(2.7) \qquad {}_{1}\Gamma^{i}_{jk} = {}_{2}\Gamma^{i}_{jk} + \gamma_j \delta^i_k + \gamma_k \delta^i_j$$

where (γ_j) is an arbitrary covector field. Applying (2.7) to (2.5) and (2.6), we may get

$$\tilde{\alpha}_j \delta^i_k - \tilde{\alpha}^i g_{jk} = \tilde{\beta}_j \delta^i_k + \gamma_j \delta^i_k - \tilde{\beta}^i g_{jk} + \gamma_k \delta^i_j \quad .$$

After lowering the index i and transvecting by g^{ij}, we get

$$(n+1)\gamma_k = 0$$

or

Lemma 4. *Two different self-recurrent* SW-O_ns, *having their adjoint Riemannian space in common, have their covariantly autoparallel lines in common if and only if covariant parts of their regular metric connections have the same coefficients.*

Then, the necessary and sufficient condition for two different self-recurrent SW-O_ns over the same Riemannian space to be covariantly projective is

(2.8) $$\tilde{\alpha}_j\delta^i_k - g_{jk}\tilde{\alpha}^i = \tilde{\beta}_j\delta^i_k - g_{jk}\tilde{\beta}^i .$$

Denoting $\tilde{\alpha}_j-\tilde{\beta}_j$ by P_j, we immediately get

$$P_j = 0$$

which means $\tilde{\alpha}_j = \tilde{\beta}_j$. But $\tilde{\alpha}_j = \alpha_a\tilde{A}^a_j$, $\tilde{\beta}_j = \beta_a\tilde{B}^a_j$ ((2.3), (2.4)). Now, we can state the additional part of Lemma 4:

Lemma 4'. *Two different self-recurrent* SW-O_ns *over the same Riemannian space are covariantly projective if and only if the images of recurrency vectors of their fundamental isomorphisms in the adjoint Riemannian space by their inverse isomorphisms respectively are the same.*

Now, let us suppose that one of these two self-recurrent SW-O_ns has its covariantly autoparallel lines as geodesics in the adjoint Riemannian space. Then the same holds for the other. We have

Theorem 2. *If the Riemannian space* M^g_n *admits two (or more) different regular symmetric tensor fields* (A_{ij}) *and* (B_{ij}), *both (all) recurrent to the metric tensor and their recurrency covectors related by*

$$\alpha_k = \beta_a\tilde{B}^a_j A^j_k$$

($\tilde{B}^a_j$ *being the inverse of* B^i_j) *and if* M^g_n *admits a tensor of type (0.2) satisfying (1.10), then* (β_j) *also satisfies such a condition. Then these two (or more)* SW-O_ns *have their covariantly autoparallel lines in common and they may serve as* π*-geodesics in* M^g_n.

3. THE POSSIBILITY OF π-COVARIANT PROJECTIVITY BETWEEN TWO DIFFERENT SW-O_ns OVER THE SAME RIEMANNIAN SPACE

Let us suppose we are given a regular tensor field of type (0.2), namely (π_{ij}), without any more presumptions about it.

If we have two different SW-O_ns over the same Riemennian space M_n, we want to answer the question: whether of not these two SW-O_ns can be π-covariantly projective, that is, to have their π-covariantly autoparallel lines in common.

Let us denote them by SW-O_n and $\overline{\text{SW-}O_n}$ and covariant parts of their metric regular connection by $"\overset{m}{\Gamma}$ and $"\overset{m}{\overline{\Gamma}}$, respectively. Let $"\overset{m}{\Gamma}$ be given by the relation

$$(3.1)\qquad "\overset{m}{\Gamma}{}^i_{jk} = \{{}^{\,i}_{jk}\} + \tilde{\alpha}_j\delta^i_k - g_{jk}\tilde{\alpha}^i .$$

If the connection $"\overset{m}{\overline{\Gamma}}{}^i_{jk}$ should be π-covariantly projective too, then, according to (0.2) it yields:

$$(3.2)\qquad "\overset{m}{\overline{\Gamma}}{}^i_{jk} = \{{}^{\,i}_{jk}\} + \tilde{\alpha}_j\delta^i_k - g_{jk}\tilde{\alpha}^i - \pi_{jk}\psi_s\overline{\pi}^{si} - \psi_j\delta^i_k$$

where (ψ_k) is an arbitrary covector field.

As we have supposed, $"\overset{m}{\overline{\Gamma}}$ should be a covariant part of regular general metric connection on $\overline{\text{SW-}O_n}$. Then, $"\overset{m}{\overline{\Gamma}}$ has to be

(a) a semi-symmetric connection,

(b) a metric connection, i.e. the metric tensor should be parallel.

Let us check condition (a) first. The torsion tensor of $"\overset{m}{\overline{\Gamma}}$ will be denoted by $"\overset{m}{\overline{T}}$.

$$(3.3)\qquad "\overset{m}{\overline{T}}{}^i_{jk} = \tfrac{1}{2}[(\tilde{\alpha}_j-\psi_j)\delta^i_k-(\tilde{\alpha}_k-\psi_k)\delta^i_j-\pi_{jk}\psi_s\overline{\pi}^{si}-\pi_{kj}\psi_s\overline{\pi}^{si}]$$

We can easily see that $\overset{m}{\overline{{}^{\pi}\Gamma}}$ is semi-symmetric if the tensor (π_{ij}) is symmetric. If the tensor (π_{ij}) is not skew--symmetric, we can always find a vector field (P_i) satisfying

$$\overset{m}{\overline{{}^{\pi}\Gamma}}{}^{i}_{jk} = \frac{1}{2}[(\tilde{a}_j-\psi_j-P_j)\delta^i_k - (\tilde{a}_k-\psi_k+P_k)\delta^i_j],$$

where (P_k) satisfies

$$P_j = \frac{1}{n-1}(\hat{\psi}_j-\psi_j) \tag{3.4}$$

$$\psi_j = \pi_{jk}\psi_s\overline{\pi}^{sk} \tag{3.5}$$

$$\hat{\psi}_j = \pi_{kj}\psi_s\overline{\pi}^{sk} \tag{3.6}$$

Then, we can state

Lemma 5. *The connection, which is* π *-projective to the covariant part of the metric regular general connection of a self-recurrent* SW-O_n *is always semi-symmetric, unless the tensor* (π_{ij}) *is skew-symmetric.*

Now, let us check condition (b). We immediately get

$$\overset{m}{\overline{{}^{\pi}\nabla}}_k g_{ij} = \pi_{ik}\overline{\psi}_j + \pi_{jk}\overline{\psi}_i + \psi_i g_{jk} + \psi_j g_{ik} \tag{3.7}$$

$\overline{\psi}_j$ means $\psi_s\overline{\pi}^{si}g_{ij}$. Since the metric tensor should be parallel, (3.7) has to vanish. Raising the index k, we get

$$\pi^{k}_{i}\overline{\psi}_j + \pi^{k}_{j}\overline{\psi}_i + \psi_j\delta^k_i + \psi_i\delta^k_j = 0 \tag{3.8}$$

Contracting the indices k, i, we get

$$\mathrm{tr}\pi\overline{\psi}_j + \psi_j + n\psi_j + \psi_j = 0$$

i.e.

$$\overline{\psi}_j = -\frac{n+2}{\mathrm{tr}\pi}\psi_j \tag{3.9}$$

which means

Lemma 6. *The connection (3.2), which is π-projective to connection* $"\overset{m}{\bar{\Gamma}}$ *is a metric connection if and only if* $\bar{\Psi}_j = -\frac{n+2}{tr\pi}\psi_j$.

Finally, if connection (3.2) ought to be a covariant metric connection of a self-recurrent $SW\text{-}O_n$, it has to have the form (3.1). According to Lemma 6. the connection $"\overset{m}{\bar{\Gamma}}$ has the form

$$"\overset{m}{\bar{\Gamma}}{}^i_{jk} = \{{}^{\,i}_{jk}\} + (\tilde{\alpha}_j - \psi_j)\delta^i_k + \frac{n+2}{tr\pi}\pi_{jk}\psi^i - g_{jk}\tilde{\alpha}^i$$

and

$$g_{jk}\tilde{\alpha}^i - \frac{n+2}{tr\pi}\pi_{jk}\psi^i = g_{jk}(\tilde{\alpha}^i - \psi^i)$$

or, shorter

$$\frac{n+2}{tr\pi}\pi_{jk}\psi^i = g_{jk}\psi^i \ .$$

Raising the index k, and contracting indices k and j, we finally get

$$\psi^i = 0 \quad \text{i.e.} \quad \psi_i = 0 \ .$$

So, the connection which is π-projective to the covariant metric connection in a self-recurrent $SW\text{-}O_n$, can never have the form (3.1), unless that π-projectivity is a projectivity itself

Theorem 3. *Two different self-recurrent* $SW\text{-}O_n$s *over the same Riemannian space can never be π-covariantly projective, unless they are projective.*

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REZIME

O KOVARIJANTNO-PROJEKTIVNIM TRANSFORMACIJAMA SAMO-REKURENTNOG SW-O_n

Predmet rada su samorekurentni Vejl-Ocukijevi prostori druge vrste. Posmatrana je mogućnost da kovarijantno autoparalelne linije u SW-O_n budu π-geodezijske u pridruženom Rimanovom prostoru, mogućnosti da dva različita SW-O_n nad istim Rimanovim prostorom budu projektivna i π-projektivna. Dokazane su i odgovarajuće teoreme.

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