Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 17,2, 129-136 (1987)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

ON PARA-ASSOCIATIVE BCC-ALGEBRAS

Izabela M. Dudek

Institute of Mathematics, Pedagogical University, Al. Zawadzkiego 13/15 42-200 Częstochowa, Poland

ABSTRACT

An algebra $(G,\cdot,0)$ of type $(2,0)$ is called a weak BCC-algebra iff $xy = yx = 0$ implies $x = y$ and if the conditions $(yz)((xy)(xz)) = 0$, $xx = 0$, $0x = x$ hold for all $x,y,z \in G$. In the paper it is proved that a weak BCC-algebra is a BCI-algebra, i.e. satisfies $((xy)xz))(zy) = 0$ and $x0 = x$ iff it is a Boolean group. It is proved also that every left (right) alternative weak BCC-algebra is a Boolean group. In the end the so-called para-associative weak BCC-algebras are considered, i.e. weak BCC-algebras with the condition: $(x_1x_2)x_3 = x_i(x_jx_k)$, where (i,j,k) is a fixed permutation of the set $\{1,2,3\}$. It is proved that these week BCC-algebras are Boolean groups.

BCK-algebras where introduced as an algebraic formulation of C.A. Meredith's BCK-implicational calculus by K. Iséki in [5] . BCK-algebras form a quasivariety of algebras amongst whose subclasses can be found the earlier implicational models of Henkin [3], algebras of sets closed under set-subtraction, and dual relatively pseudocomplemented upper semilattices.

AMS Mathematics Subject Classification (1980): Primary 03G25, Secondary 08B99
Key words and phrases: BCC-algebra, weak BCC-algebra, BCI-algebra, para-associativity.

K. Iséki posed an interesting problem (solved by A. Wroński in [8]) whether the class of BCK-algebras is a variety. In connection with this problem Y. Komori introduced in [6] the notion of BCC-algebras and proved (using some Gentzen-type system LC) that the class of all BCC-algebras is not a variety, but the variety genarated by BCC-algebras, that is, the smallest vatiety containing the class of all BCC-algebras, is finitely based [7].

In this note we shall consider so-called weak BCC-algebras which are a generalization of BCC-algebras and give some necessary and sufficient conditions for a weak BCC-algebra to be a Boolean group.

1.

First we shall give some basic definitions and results.

By a *BCC-algebra* we mean a non-empty set G together with a binary multiplication denoted by juxtaposition and a distinguished element 0 such that the following axioms are satisfied for all $x,y,z \in G$:

(1) $$(yz)((xy)(xz)) = 0,$$

(2) $$xx = 0,$$

(3) $$x0 = x,$$

(4) $$0x = x,$$

(5) $$xy = yx = 0 \quad \text{implies} \quad x = y.$$

By a BCI-algebra we mean an algebra $(G; \cdot, 0)$ of type (2,0) in which conditions (2), (5) and

(6) $$((xy)(xz))(zy) = 0,$$

(7) $$(x(xy))y = 0,$$

(8) $$x0 = x$$

are satisfied.

If a BCI-algebra $(G,\cdot,0)$ satisfies also $0x = 0$, then it is called a *BCK-algebra*.

The aobove axiom systems are not independent. One can prove (cf. [1]) that the class of all *BCI-algebras is uniquely determined by (5), (6) and (8). BCK-algebras are determined by (5), (6), (8) and $0x = 0$.* Also axioms of (weak) BCC--algebras are dependent. Indeed, putting $x = y = 0$ in (1) and using (4), we obtain (2).

Y. Komori noticed in [6] that if we exchange (1) for

$$(xy)((yz)(xz)) = 0, \tag{9}$$

then we obtain the axiom system of BCK-algebras (but dual form).

Similarly, if we consider a genereal algebra $(G,\cdot,0)$ of type (2,0) which satisfies (1), (2), (4), (5) and if we exchange (1) for (9), then we obtain the dual form of the axioms system of BCI-algebras which are a generalization of BCK--algebras. Hence, a general algebra $(G,\cdot,0)$ of type (2) with conditions (1), (2), (4) and (5) is called a weak BCC-algebra.

A simple example of a weak BCC-algebra is a Boolean group or an algebra $(G,*,0)$ with $x * y = y - x$, where $(G,+,0)$ is an Abelian group. These algebras are not BCC-algebras.

A groupoid $(G,\cdot)$ is called a *para-associative groupoid of type* (i,j,k) (cf. [2]) or an (i,j,k)-*associative groupoid,* if

$$(x_1\cdot x_2)\cdot x_3 = x_i\cdot(x_j\cdot x_k)$$

for all x_1, x_2, $x_3 \in G$, where (i,j,k) is a fixed permutation of 1, 2 and 3. It is clear that *a para-associative groupoid of type* (1,2,3) is a semigroup. Every (1,3,2)-associative groupoid is left alternative,every (2,1,3)-associative - right alternative. The class od all flexible groupoids is contained in the class of all (2,3,1)-associative groupoids.

2.

First we shall prove a simple but useful lemma.

Lemma. *A weak BCC-algebra is a BCI-algebra iff it is Abelian.*

Proof. An Abelian weak BCC-algebra is obviously a BCI-algebra. On the other hand, if a weak BCC-algebra $(G,\cdot,0)$ is a BCI-algebra, then $x = 0x = x0$ for all $x \in G$. Hence by the Lemma from [1] we obtain $xy = yx$, which finishes our proof.

As it is well-known (see [4]) a BCI-algebra is a Boolean group iff it is associative or iff it satisfies the identity $x = 0x$. Hence as a simple consequence of the above lemma we obtain

Corollary 1. *A weak BCC-algebra is a BCI-algebra iff it is a Boolean group.*

Remark 1. From (3) it follows that a Boolean group is not a BCC-algebra. Moreover, from (3) and (4) follows that a BCC-algebra is a BCK-algebra iff it is trivial, i.e. iff it has only one element.

Proposition 1. *Every left (right) alternative weak BCC-algebra is a Boolean group.*

Proof. If a weak BCC-algebra $(G,\cdot,0)$ is left alternative, then $(yx)x=y(xx)$ for all $x,y \in G$. Hence $x = (x0)0 = x(00) = x0$. Thus putting $z = x$ in (1), we obtain $0 = (yx)((xy)(xx)) = (yx)(xy)$, which proves that $(G,\cdot,0)$ is Abelian.

Similarly, if $(G,\cdot,0)$ is right alternative, i.e. if $(xx)y = x(xy)$, then $y = x(xy)$ and $x0 = x$. Since (1) for $z = 0$

implies $y((xy)x) = 0$, then, putting in this equation $y = xz$, we get $0 = (xz((x(xz))x) = (xz)(zx)$, which proves that also in this case $(G,\cdot,0)$ is Abelian. But by the Lemma an Abelian weak BCC-algebra is a BCI-algebra, i.e. $(G,\cdot,0)$ is a Boolean group (Corollary 1).

Proposition 2. *A weak BCC-algebra with the identity* $(xy)x = y$ *is a Boolean group.*

Proof. If a weak BCC-algebra $(G,\cdot,0)$ satisfies $(xy)x = y$, then $y = y0$ and for every $a,b \in G$ the equation $bx = a$ is uniquely solvable. Indeed, an element $x = ab$ satisfies the equation $xa = b$ and $bx = a$, since $(ab)a \in b$ and $a = (xa)x = bx$. Moreover $bx = bz$ gives $x = z$, since $x = (bx)b = (bz)b = z$. Thus for every $a,b \in G$ the solution $x \in G$ of the equation $bx = a$ exists and is uniquely determined. Hence from (1) we have $(xy)(xz) = yz$ and $x(xy) = (x0)(xy) = 0y = (xx)y$, which shows that this weak BCC-algebra is right alternative. By Proposition 1 it is a Boolean group.

Corollary 2. *If a weak BCC-algebra* $(G,\cdot,0)$ *is a quasigroup, then it is a (Boolean) group iff* $x0 = x$ *for all* $x \in G$, *i.e. iff it has a neutral element.*

Proof. If a weak BCC-algebra $(G,\cdot,0)$ is a quasigroup and 0 is its neutral element, then from (1) we get $yz = (xy)(xz)$ and $y = y0 = (xy)(x0) = (xy)x$, which finishes the first part of the proof.

The second part is obvious.

In the same manner as Proposition 2, we can prove

Proposition 3. *A weak BCC-algebra with the identity* $x(yx) = y$ *is a Boolean group.*

We are now in a position to state the full characterization of para-associative weak BCC-algebras.

Proposition 4. *Every para-associative weak BCC-algebra is a Boolean group.*

Proof. We shall consider six cases of the para-associativity.

i) The case of (1,2,3)-associativity.

Since the multiplication in this case is associative, then from (1) we get

$$0 = (yz)((xy)(xz)) = ((yz)x)(y(xz))$$

and

$$0 = (yx)((zy)(zx)) = (y(xz))((yz)x),$$

which gives (by (5)) $(yz)x = y(xz)$. Hence a (1,2,3)-associative weak BCC-algebra is also (1,3,2)-associative.

ii) The case of (1,3,2)-associativity immeadiately follows from Proposition 1, since every (1,3,2)-associative weak BCC-algebra is left alternative.

iii) The case of (2,1,3)-associativity also follows from Proposition 1 since in this case a weak BCC-algebra is right alternative.

iv) In the case of (2,3,1)-associativity, we have $(xy)z = y(zx)$ for all $x,y \in G$. Hence $x = (00)x = 0(x0) = x0$ and $(xy)x = y(xx) = y0 = y$ which proves (by Proposition 2) that this weak BCC-algebra is a Boolean group.

v) The case of (3,1,2)-associativity.

Putting $x = 0$ in $(xy)z = z(xy)$, we obtain $yz = zy$, which implies (by the Lemma) that this weak BCC-algebra is Abelian. It is obviously a Boolean group (Corollary 1).

vi) The case of (3,2,1)-associativity.

As in the previous case putting $x = 0$ in $(xy)z = z(yx)$, we get $yz = zy$, since

$$z = 0z = (xx)z = z(xx) = z0$$

and

$$yz = (0y)z = z(y0) = zy.$$

Remark 2. It is obvious that every Boolean group satisfies all the conditions given in the above propositions. Therefore, these conditions are necessary and sufficient for a weak BCC-algebra to be a Boolean group.

REFERENCES

[1] Dudek W.A., *On the axioms system for BCI-algebras (to appear)*.

[2] Глейхгевихт Б., *Парaассоциативные кольца и алгебры*, *Acta Univ. Wratislaviensis 58 (1967), 21 - 35.*

[3] Henkin L., *An algebraic characterization of quantifiers, Fund. Math., 37 (1950), 63 - 74.*

[4] Hu Q.P., Iséki K., *On BCI-algebras satisfying (x*y)*z = = x*(y*z), Math. Sem., Notes Kobe Univ., 8 (1980), 123 - 130.*

[5] Iséki K., *An algebra related with a propositional calculus, Proc. Japan Acad. Ser. A Math. Sci., 42 (1966), 26 - 29.*

[6] Komori Y., *The class of BCC-algebras is not a variety, Math. Japon., 29 (1984), 391 - 394.*

[7] Komori Y., *The variety generated by BCC-algebras is finitely based, Reports Fac. Sci. Shizuoka Univ., 17 (1983), 13 - 16.*

[8] Wroński A., *BCK-algebras do not form a variety, Math. Japon., 28 (1983), 211 - 213.*

REZIME

O PARA-ASOCIJATIVNIM BCC-ALGEBRAMA

Algebra $(G,\cdot,0)$ tipa $(2,0)$ se naziva slaba BCC-algebra ako i samo ako $xy = yx = 0$ implicira $x = y$ i ako uslovi $(yz)((xy)(xz)) = 0$ $xx = 0$, $0x = x$ važe sa sve $x,y,z \in G$. U ovom radu je dokazano da je jedna slaba BCC-algebra i BCI-algebra odnosno važi $((xy)xz))(zy)=0$ i $x0=x$ ako i samo ako je ona jedna Bulova grupa. Dokazano je takodje da je svaka levo (desno) alternativna slaba BCC-algebra i jedna Bulova grupa. Posmatrane su takozvane para-asocijativne slabe BCC-algebre odnosno slabe BCC-algebre sa uslovom: $(x_1x_2)x_3 = x_i(x_jx_k)$, gde je (i,j,k) fiksirana permutacija skupa $\{1,2,3\}$. Dokazano je da su ove BCC-algebre Bulove grupe.

Received by the editors March 16,1987.